Differential Equations for Engineers

Lecture Notes for Coursera

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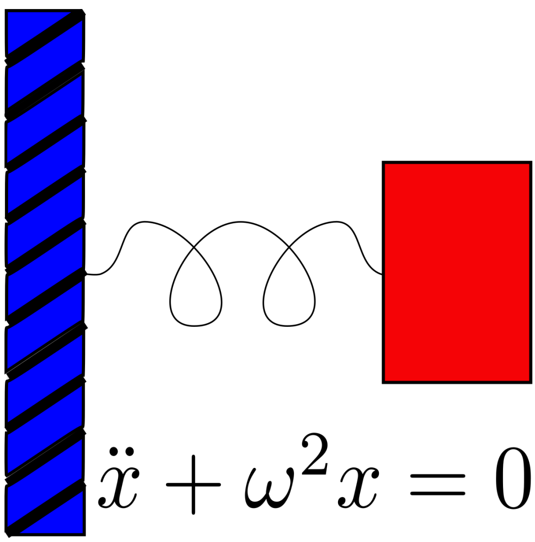

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Preface

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These are my lecture notes for my online Coursera course, Differential Equations for Engineers. I have divided these notes into chapters called Lectures, with each Lecture corresponding to a video on Coursera. I have also uploaded all my Coursera videos to YouTube, and links are placed at the top of each Lecture.

There are problems at the end of each lecture chapter and I have tried to choose problems that exemplify the main idea of the lecture. Students taking a formal university course in differential equations will usually be assigned many more additional problems, but here I follow the philosophy that less is more. I give enough problems for students to solidify their understanding of the material, but not too many problems that students feel overwhelmed and drop out. I do encourage students to attempt the given problems, but if they get stuck, full solutions can be found in the Appendix.

There are also additional problems at the end of coherent sections that are given as practice quizzes on the Coursera platform. Again, students should attempt these quizzes on the platform, but if a student has trouble obtaining a correct answer, full solutions are also found in the Appendix.

Students who take this course are expected to know single-variable differential and integral calculus. Some knowledge of complex numbers, matrix algebra and vector calculus is required for parts of this course. Students missing this latter knowledge can find the necessary material in the Appendix.

Jeffrey R. Chasnov
Hong Kong
January 2019

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Lecture 1 | Introduction to differential equations

View this lecture on YouTube

A differential equation is an equation for a function containing derivatives of that function. For example, the differential equations for an *RLC* circuit, a pendulum, and a diffusing dye are given by

$$
\begin{aligned}
&L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \mathcal{E}_0 \cos \omega t, && (RLC \text{ circuit equation})\\
&ml\frac{d^2\theta}{dt^2} + cl\frac{d\theta}{dt} + mg\sin\theta = F_0 \cos \omega t, && (\text{pendulum equation})\\
&\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). && (\text{diffusion equation})
\end{aligned}
$$

These are second-order differential equations, categorized according to the highest order derivative.

The *RLC* circuit equation (and pendulum equation) is an ordinary differential equation, or ode, and the diffusion equation is a partial differential equation, or pde. An ode is an equation for a function of a single variable and a pde for a function of more than one variable. A pde is theoretically equivalent to an infinite number of odes, and numerical solution of nonlinear pdes may require supercomputer resources.

The *RLC* circuit and the diffusion equation are linear and the pendulum equation is nonlinear. In a linear differential equation, the unknown function and its derivatives appear as a linear polynomial. For instance, the general linear third-order ode, where $y = y(x)$ and primes denote derivatives with respect to x, is given by

$$a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x),$$

where the a and b coefficients can be any function of x. The pendulum equation is nonlinear because of the term $\sin\theta$, where $\theta = \theta(t)$ is the unknown function. Making the small angle approximation, $\sin\theta \approx \theta$, the pendulum equation becomes linear.

The simplest type of ode can be solved by integration. For example, a mass such as Newton's apocryphal apple, falls downward with constant acceleration, and satisfies the differential equation

$$\frac{d^2x}{dt^2} = -g.$$

With initial conditions specifying the initial height of the mass x_0 and its initial velocity u_0, the solution obtained by straightforward integration is given by the well-known high school physics equation

$$x(t) = x_0 + u_0 t - \frac{1}{2}gt^2.$$

Practice Quiz | Classify differential equations

1. By checking all that apply, classify the following differential equation:

$$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} = 0$$

a) first order

b) second order

c) third order

d) ordinary

e) partial

f) linear

g) nonlinear

2. By checking all that apply, classify the following differential equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi}$$

a) first order

b) second order

c) ordinary

d) partial

e) linear

f) nonlinear

3. By checking all that apply, classify the following differential equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

a) first order

b) second order

c) ordinary

d) partial

e) linear

f) nonlinear

4. By checking all that apply, classify the following differential equation:

$$a\frac{d^2 x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

a) first order

b) second order

c) ordinary

d) partial

e) linear

f) nonlinear

5. By checking all that apply, classify the following differential equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

a) first order

b) second order

c) ordinary

d) partial

e) linear

f) nonlinear

Solutions to the Practice quiz

Week I

First-Order Differential Equations

In this week's lectures, we discuss first-order differential equations. We begin by explaining the Euler method, which is a simple numerical method for solving an ode. Not all first-order differential equations have an analytical solution, so it is useful to understand a basic numerical method. Then the analytical solution methods for separable and linear equations are explained. We follow the discussion of each theory with some simple examples. Finally, three real-world applications of first-order equations and their solutions are presented: compound interest, terminal velocity of a falling mass, and the resistor-capacitor electrical circuit.

Lecture 2 | Euler method

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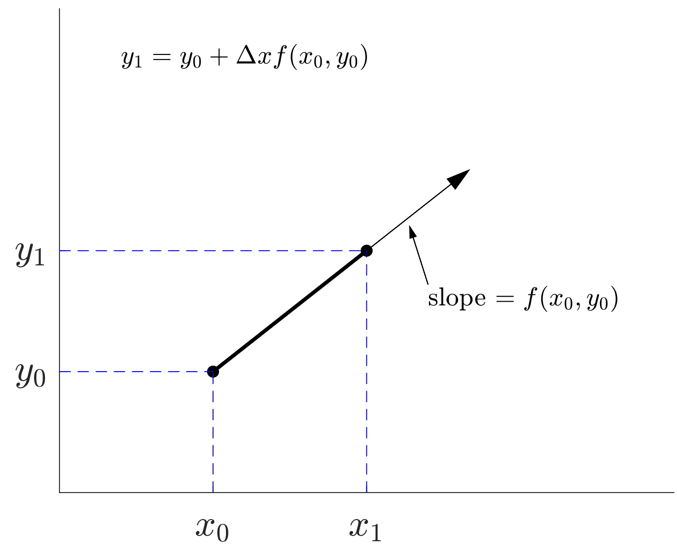

Sometimes there is no analytical solution to a first-order differential equation and a numerical solution must be sought. The first-order differential equation $dy/dx = f(x, y)$ with initial condition $y(x_0) = y_0$ provides the slope $f(x_0, y_0)$ of the tangent line to the solution curve $y = y(x)$ at the point (x_0, y_0). With a small step size $\Delta x = x_1 - x_0$, the initial condition (x_0, y_0) can be marched forward to (x_1, y_1) along the tangent line using Euler's method (see figure):

$$y_1 = y_0 + \Delta x f(x_0, y_0).$$

This solution (x_1, y_1) then becomes the new initial condition and is marched forward to (x_2, y_2) along a newly determined tangent line with slope given by $f(x_1, y_1)$. For small enough Δx, the numerical solution converges to the unique solution, when such a solution exists.

There are better numerical methods than the Euler method, but the basic principle of marching the solution forward remains the same.

Problems for Lecture 2

1. The Euler method for solving the differential equation $dy/dx = f(x, y)$ can be rewritten in the form

$$k_1 = \Delta x f(x_n, y_n), \quad y_{n+1} = y_n + k_1,$$

and is called a first-order Runge-Kutta method. More accurate second-order Runge-Kutta methods have the form

$$k_1 = \Delta x f(x_n, y_n), \quad k_2 = \Delta x f(x_n + \alpha \Delta x, y_n + \beta k_1), \quad y_{n+1} = y_n + a k_1 + b k_2.$$

Some analysis (not shown here) on the second-order Runge-Kutta methods results in the constraints

$$a + b = 1, \qquad \alpha b = \beta b = 1/2.$$

Write down the second-order Runge-Kutta methods corresponding to (i) $a = b$, and (ii) $a = 0$. These specific second-order Runge-Kutta methods are called the modified Euler method and the midpoint method, respectively.

Solutions to the Problems

Lecture 3 | Separable first-order equations

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A first-order ode is separable if it can be written in the form

$$g(y)\frac{dy}{dx} = f(x), \quad y(x_0) = y_0,$$

where the function $g(y)$ is independent of x and $f(x)$ is independent of y. Integration from x_0 to x results in

$$\int_{x_0}^{x} g(y(x))y'(x)\,dx = \int_{x_0}^{x} f(x)\,dx.$$

The integral on the left can be transformed by substituting $u = y(x)$, $du = y'(x)dx$, and changing the lower and upper limits of integration to $y(x_0) = y_0$ and $y(x) = y$. Therefore,

$$\int_{y_0}^{y} g(u)\,du = \int_{x_0}^{x} f(x)\,dx,$$

which can often yield an analytical expression for $y = y(x)$ if the integrals can be done and the resulting algebraic equation can be solved for y.

A simpler procedure that yields the same result is to treat dy/dx as a fraction. Multiplying the differential equation by dx results directly in

$$g(y)\,dy = f(x)\,dx,$$

which is what we call a separated equation with a function of y times dy on one side, and a function of x times dx on the other side. This separated equation can then be integrated directly over y and x.

Problems for Lecture 3

1. Put the following equation in separated form. Do not integrate.

a) $\dfrac{dy}{dx} = \dfrac{x^2y - 4y}{x+4}$

b) $\dfrac{dy}{dx} = \sec(y)e^{x-y}(1+x)$

c) $\dfrac{dy}{dx} = \dfrac{xy}{(x+1)(y+1)}$

d) $\dfrac{d\theta}{dt} + \sin\theta = 0$

Solutions to the Problems

Lecture 4 | Separable first-order equation (example)

[View this lecture on YouTube]

Example: Solve $y' + y^2 \sin x = 0, \quad y(0) = 1.$

We first manipulate the differential equation to the form

$$\frac{dy}{dx} = -y^2 \sin x$$

and then treat dy/dx as if it was a fraction to separate variables:

$$\frac{dy}{y^2} = -\sin x \, dx.$$

We then integrate the right side from x equals 0 to x and the left side from y equals 1 to y. We obtain

$$\int_1^y \frac{dy}{y^2} = -\int_0^x \sin x \, dx.$$

Integrating, we have

$$-\frac{1}{y}\bigg|_1^y = \cos x\bigg|_0^x,$$

or

$$1 - \frac{1}{y} = \cos x - 1.$$

Solving for y, we obtain the solution

$$y = \frac{1}{2 - \cos x}.$$

Problems for Lecture 4

1. Solve the following separable first-order equations.

a) $dy/dx = 4x\sqrt{y}$, with $y(0) = 1$.

b) $dx/dt = x(1-x)$, with $x(0) = x_0$ and $0 < x_0 < 1$.

Solutions to the Problems

Practice Quiz | Separable first-order odes

1. The solution of $y' = \sqrt{xy}$ with initial value $y(1) = 0$ is given by

a) $y(x) = \frac{1}{9}(x^{1/2} - 1)^2$

b) $y(x) = \frac{1}{9}(x - 1)^2$

c) $y(x) = \frac{1}{9}(x^{3/2} - 1)^2$

d) $y(x) = \frac{1}{9}(x^2 - 1)^2$

2. The solution of $y^2 - xy' = 0$ with initial value $y(1) = 1$ is given by

a) $y(x) = \dfrac{1}{1 - \ln x}$

b) $y(x) = \dfrac{1}{1 - 2\ln x}$

c) $y(x) = \dfrac{1}{1 + \ln x}$

d) $y(x) = \dfrac{1}{1 + 2\ln x}$

3. The solution of $y' + (\sin x)y = 0$ with initial value $y(\pi/2) = 1$ is given by

a) $y(x) = e^{\sin x}$

b) $y(x) = e^{\cos x}$

c) $y(x) = e^{1-\sin x}$

d) $y(x) = e^{1-\cos x}$

Solutions to the Practice quiz

Lecture 5 | Linear first-order equations

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A linear first-order differential equation with initial condition can be written in standard form as

$$\frac{dy}{dx} + p(x)y = g(x), \qquad y(x_0) = y_0. \tag{5.1}$$

All linear first-order differential equations can be integrated using an integrating factor μ. We multiply the differential equation by the yet unknown function $\mu = \mu(x)$ to obtain

$$\mu(x)\left[\frac{dy}{dx} + p(x)y\right] = \mu(x)g(x);$$

and then require $\mu(x)$ to satisfy the differential equation

$$\mu(x)\left[\frac{dy}{dx} + p(x)y\right] = \frac{d}{dx}[\mu(x)y]. \tag{5.2}$$

The unknown function $\mu(x)$ is called an integrating factor because the resulting differential equation, $\frac{d}{dx}[\mu(x)y] = \mu(x)g(x)$, can be directly integrated. Using $y(x_0) = y_0$ and choosing $\mu(x_0) = 1$, we have $\mu(x)y - y_0 = \int_{x_0}^{x} \mu(x)g(x)\,dx$; or after solving for $y = y(x)$,

$$y(x) = \frac{1}{\mu(x)}\left(y_0 + \int_{x_0}^{x} \mu(x)g(x)\,dx\right). \tag{5.3}$$

To determine $\mu(x)$, we expand (5.2) using the product rule to obtain to yield

$$\mu\frac{dy}{dx} + p\mu y = \frac{d\mu}{dx}y + \mu\frac{dy}{dx};$$

and upon canceling terms, we obtain the differential equation

$$\frac{d\mu}{dx} = p(x)\mu, \quad \mu(x_0) = 1.$$

This equation is separable and can be easily integrated to obtain

$$\mu(x) = \exp\left(\int_{x_0}^{x} p(x)\,dx\right). \tag{5.4}$$

Equations (5.3) and (5.4) together solve the first-order linear equation given by (5.1).

Problems for Lecture 5

1. Write the following linear equations in standard form.

a) $x\dfrac{dy}{dx} + y = \sin x$;

b) $\dfrac{dy}{dx} = x - y$.

2. Consider the nonlinear differential equation $dx/dt = x(1-x)$. By defining $z = 1/x$, show that the resulting differential equation for z is linear.

Solutions to the Problems

Lecture 6 | Linear first-order equation (example)

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Example: Solve $\frac{dy}{dx} + 2y = e^{-x}$, *with* $y(0) = 3/4$.

Note that this equation is not separable. With $p(x) = 2$ and $g(x) = e^{-x}$, we have

$$\begin{aligned}\mu(x) &= \exp\left(\int_0^x 2\,dx\right)\\ &= e^{2x},\end{aligned}$$

and

$$y(x) = e^{-2x}\left(\frac{3}{4} + \int_0^x e^{2x}e^{-x}\,dx\right).$$

Performing the integration, we obtain

$$y(x) = e^{-2x}\left(\frac{3}{4} + (e^x - 1)\right),$$

which can be simplified to

$$y(x) = e^{-x}\left(1 - \frac{1}{4}e^{-x}\right).$$

Problems for Lecture 6

1. Solve the following linear odes.

a) $\dfrac{dy}{dx} = x - y, \quad y(0) = -1;$

b) $\dfrac{dy}{dx} = 2x(1 - y), \quad y(0) = 0.$

Solutions to the Problems

Practice Quiz | Linear first-order odes

1. The solution of $(1+x^2)y' + 2xy = 2x$ with initial value $y(0) = 0$ is given by

a) $y(x) = \dfrac{x}{1+x}$

b) $y(x) = \dfrac{x}{1+x^2}$

c) $y(x) = \dfrac{x^2}{1+x}$

d) $y(x) = \dfrac{x^2}{1+x^2}$

2. The solution of $x^2 y' = 1 - 2xy$ with initial value $y(1) = 2$ is given by

a) $y(x) = \dfrac{1+x}{x}$

b) $y(x) = \dfrac{1+x}{x^2}$

c) $y(x) = \dfrac{1+x^2}{x}$

d) $y(x) = \dfrac{1+x^2}{x^2}$

3. The solution of $y' + \lambda y = a$ with initial value $y(0) = 0$ and $\lambda > 0$ is given by

a) $y(x) = a(1 - e^{\lambda x})$

b) $y(x) = a(1 - e^{-\lambda x})$

c) $y(x) = \dfrac{a}{\lambda}(1 - e^{\lambda x})$

d) $y(x) = \dfrac{a}{\lambda}(1 - e^{-\lambda x})$

Solutions to the Practice quiz

Lecture 7 | Application: compound interest

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The compound interest equation arises in many different engineering problems, and we consider it here as it relates to an investment. Let $S(t)$ be the value of the investment at time t, and let r be the annual interest rate compounded after every time interval Δt. Let k be the annual deposit amount (a negative value indicates a withdrawal), and suppose that a fixed amount is deposited after every time interval Δt. The value of the investment at the time $t + \Delta t$ is then given by

$$S(t + \Delta t) = S(t) + (r\Delta t)S(t) + k\Delta t,$$

where at the end of the time interval Δt, $r\Delta t S(t)$ is the amount of interest credited and $k\Delta t$ is the amount of money deposited.

Rearranging the terms of this equation to exhibit what will soon become a derivative, we have

$$\frac{S(t + \Delta t) - S(t)}{\Delta t} = rS(t) + k.$$

The equation for continuous compounding of interest and continuous deposits is obtained by taking the limit $\Delta t \to 0$. The resulting differential equation is

$$\frac{dS}{dt} = rS + k,$$

which can solved with the initial condition $S(0) = S_0$, where S_0 is the initial capital. The differential equation is linear and the standard form is $dS/dt - rS = k$, so that the integrating factor is given by

$$\mu(t) = e^{-rt}.$$

The solution is therefore

$$\begin{aligned} S(t) &= e^{rt}\left(S_0 + \int_0^t ke^{-rt}dt\right) \\ &= S_0e^{rt} + \frac{k}{r}e^{rt}\left(1 - e^{-rt}\right), \end{aligned}$$

where the first term on the right-hand side comes from the initial invested capital, and the second term comes from the deposits (or withdrawals). Evidently, compounding results in the exponential growth of an investment.

Problems for Lecture 7

1. Suppose a 25 year-old plans to set aside a fixed amount each year until retirement at age 65. How much must he/she save each year to have $1,000,000 at retirement? Assume a 6% annual return on the saved money compounded continuously. Out of the $1,000,000, approximately how much was saved, and how much was earned on the investment?

2. A home buyer can afford to spend no more than $1500 per month on mortgage payments. Suppose that the annual interest rate is 4% and that the term of the mortgage is 30 years. Assume that interest is compounded continuously and that payments are also made continuously. Determine the maximum amount that this buyer can afford to borrow.

Solutions to the Problems

Lecture 8 | Application: terminal velocity

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Using Newton's law, we model a mass m free falling under gravity but with air resistance. We assume that the force of air resistance is proportional to the speed of the mass and opposes the direction of motion. We define the x-axis to point in the upward direction, opposite the force of gravity. Near the surface of the Earth, the force of gravity is approximately constant and is given by $-mg$, with $g = 9.8\,\mathrm{m/s^2}$ the usual gravitational acceleration. The force of air resistance is modeled by $-kv$, where v is the vertical velocity of the mass and k is a positive constant. When the mass is falling, $v < 0$ and the force of air resistance is positive, pointing upward and opposing the motion. The total force on the mass is therefore given by $F = -mg - kv$. With $F = ma$ and $a = dv/dt$, we obtain the differential equation

$$m\frac{dv}{dt} = -mg - kv.$$

The terminal velocity v_∞ of the mass is defined as the asymptotic velocity after air resistance balances the gravitational force. When the mass is at terminal velocity, $dv/dt = 0$ so that

$$v_\infty = -\frac{mg}{k}.$$

The approach to the terminal velocity of a mass initially at rest is obtained by solving the differential equation with initial condition $v(0) = 0$. The equation is linear and the standard form is $dv/dt + (k/m)v = -g$, so that the integrating factor is

$$\mu(t) = e^{kt/m},$$

and the solution to the differential equation is

$$\begin{aligned} v(t) &= e^{-kt/m}\int_0^t e^{kt/m}(-g)\,dt \\ &= -\frac{mg}{k}\left(1 - e^{-kt/m}\right). \end{aligned}$$

Therefore, $v = v_\infty\left(1 - e^{-kt/m}\right)$, and v approaches v_∞ as the exponential term decays to zero.

Problems for Lecture 8

1. A male skydiver of mass $m = 100\,\text{kg}$ with his parachute closed may attain a terminal speed of 200 km/hr. How long does it take him to attain one-half his terminal speed, and how long to attain 95%?

Solutions to the Problems

Lecture 9 | Application: RC circuit

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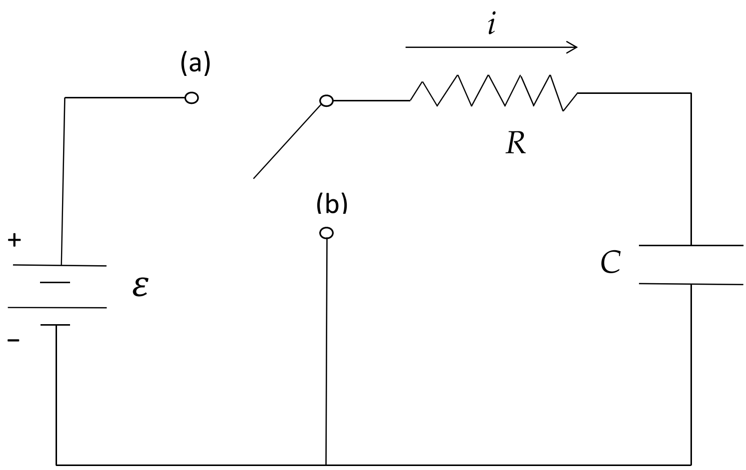

Consider a resister R and a capacitor C connected in series as shown in the figure. A battery connects to this circuit by a switch, stepping up the voltage by $\mathcal{E}$. Initially, there is no charge on the capacitor. When the switch is thrown to (a), the battery connects and the capacitor charges. When the switch is thrown to (b), the battery disconnects and the capacitor discharges, with energy dissipated in the resister. Here, we determine the voltage drop across the capacitor during charging and discharging.

The equations for the voltage drops across a capacitor and a resister are given by

$$V_C = q/C, \quad V_R = iR, \tag{9.1}$$

where C is the capacitance and R is the resistance. The charge q and the current i are related by

$$i = \frac{dq}{dt}. \tag{9.2}$$

Kirchhoff's voltage law states that the emf $\mathcal{E}$ in any closed loop is equal to the sum of the voltage drops in that loop. Applying Kirchhoff's voltage law when the switch is thrown to a results in

$$V_R + V_C = \mathcal{E}. \tag{9.3}$$

Using (9.1) and (9.2), the voltage drop across the resister can be written in terms of the voltage drop across the capacitor as

$$V_R = RC\frac{dV_C}{dt},$$

and (9.3) can be rewritten to yield the linear first-order differential equation for V_C given by

$$\frac{dV_C}{dt} + V_C/RC = \mathcal{E}/RC, \tag{9.4}$$

with initial condition $V_C(0) = 0$.

The integrating factor for this equation is

$$\mu(t) = e^{t/RC},$$

and (9.4) integrates to

$$V_C(t) = e^{-t/RC} \int_0^t (\mathcal{E}/RC) e^{t/RC} dt,$$

with solution

$$V_C(t) = \mathcal{E} \left(1 - e^{-t/RC}\right).$$

The voltage starts at zero and rises exponentially to $\mathcal{E}$, with characteristic time scale given by RC.

When the switch is thrown to b, application of Kirchhoff's voltage law results in

$$V_R + V_C = 0,$$

with corresponding differential equation

$$\frac{dV_C}{dt} + V_C/RC = 0.$$

Here, we assume that the capacitance is initially fully charged so that $V_C(0) = \mathcal{E}$. The solution, then, during the discharge phase is given by

$$V_C(t) = \mathcal{E} e^{-t/RC}.$$

The voltage starts at $\mathcal{E}$ and decays exponentially to zero, again with characteristic time scale given by RC.

Problems for Lecture 9

1. Determine the current in an *RC* circuit during charging and discharging.

Solutions to the Problems

Practice Quiz | Applications

1. Suppose a 25 year-old plans to set aside a fixed amount each year until retirement at age 65. Approximately, how much total must they have set aside to have \$1,000,000 at retirement? Assume a 10% annual return on the saved money compounded continuously.

a) \$75,000

b) \$150,000

c) \$225,000

d) \$300,000

2. A male skydiver of mass 100 kg with his parachute closed may attain a terminal speed of 200 km/hr. How long does it take him to attain a speed of 150 km/hr?

a) 1 s

b) 4 s

c) 8 s

d) 15 s

3. An RC circuit consists of a resistor (3000 Ω) and a capacitor (0.001 F), where Ω is the ohm with units $\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3} \cdot \mathrm{A}^{-2}$, F is the Faraday with units $\mathrm{s}^4 \cdot \mathrm{A}^2 \cdot \mathrm{m}^{-2} \cdot \mathrm{kg}^{-1}$, and where kg is kilogram, m is meters, s is seconds, and A is amps. How long does it take for the capacitor to charge to 95% of the battery voltage?

a) 1 s

b) 5 s

c) 9 s

d) 14 s

Solutions to the Practice quiz

Week II

Homogeneous Linear Differential Equations

This week's lectures are on second-order differential equations. We begin by generalizing the Euler numerical method to a second-order equation, to show how numerical solutions can be obtained for equations that have no analytical solutions. We then develop two theoretical concepts used for linear equations: the principle of superposition, and the Wronskian. Armed with these concepts, we can then find analytical solutions to a homogeneous second-order ode with constant coefficients. We make use of an exponential ansatz, and convert the differential equation to a quadratic equation called the characteristic equation of the ode. The characteristic equation may have real or complex roots and we discuss the solutions for these different cases.

Lecture 10 | Euler method for higher-order odes

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In practice, most higher-order odes are solved numerically. For example, consider the general second-order ode given by

$$\ddot{x} = f(t, x, \dot{x}).$$

Here, we adopt the widely used physics notation $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$. The dot notation is used to represent only time derivatives and can be applied to any dependent variable that is a function of time.

To solve numerically, we convert the second-order ode to a pair of first-order odes. Define $u = \dot{x}$, and write the first-order system as

$$\dot{x} = u, \tag{10.1}$$

$$\dot{u} = f(t, x, u). \tag{10.2}$$

The first ode, (10.1), gives the slope of the tangent line to the curve $x = x(t)$. The second ode, (10.2), gives the slope of the tangent line to the curve $u = u(t)$. Beginning at the initial values $(x, u) = (x_0, u_0)$ at the time $t = t_0$, we move along the tangent lines to determine $x_1 = x(t_0 + \Delta t)$ and $u_1 = u(t_0 + \Delta t)$:

$$\begin{aligned} x_1 &= x_0 + \Delta t u_0, \\ u_1 &= u_0 + \Delta t f(t_0, x_0, u_0). \end{aligned}$$

The values x_1 and u_1 at the time $t_1 = t_0 + \Delta t$ are then used as new initial values to march the solution forward to time $t_2 = t_1 + \Delta t$. When a unique solution of the ode exists, the numerical solution converges to this unique solution as $\Delta t \to 0$.

Problems for Lecture 10

1. Write the second-order ode, $\ddot{\theta} + \dot{\theta}/q + \sin\theta = f\cos\omega t$, as a system of two first-order odes.

2. Write down the second-order Runge-Kutta modified Euler method (predictor-corrector method) for the following system of two first-order odes:

$$\dot{x} = f(t, x, y), \quad \dot{y} = g(t, x, y).$$

Solutions to the Problems

Lecture 11 | The principle of superposition

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Consider the homogeneous linear second-order ode given by

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0;$$

and suppose that $x = X_1(t)$ and $x = X_2(t)$ are solutions. We consider a linear combination of X_1 and X_2 by letting

$$x = c_1 X_1(t) + c_2 X_2(t),$$

with c_1 and c_2 constants. The *principle of superposition* states that x is also a solution to the homogeneous ode. To prove this, we compute

$$\begin{aligned}
\ddot{x} + p\dot{x} + qx &= c_1\ddot{X}_1 + c_2\ddot{X}_2 + p\left(c_1\dot{X}_1 + c_2\dot{X}_2\right) + q\left(c_1X_1 + c_2X_2\right) \\
&= c_1\left(\ddot{X}_1 + p\dot{X}_1 + qX_1\right) + c_2\left(\ddot{X}_2 + p\dot{X}_2 + qX_2\right) \\
&= c_1 \times 0 + c_2 \times 0 \\
&= 0,
\end{aligned}$$

since X_1 and X_2 were assumed to be solutions of the homogeneous ode. We have therefore shown that any linear combination of solutions to the homogeneous linear second-order ode is also a solution.

Problems for Lecture 11

1. Consider the inhomogeneous linear second-order ode given by

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t);$$

and suppose that $x = x_h(t)$ is a solution of the homogeneous equation and $x = x_p(t)$ is a solution of the inhomogeneous equation. Prove that $x = x_h(t) + x_p(t)$ is a solution of the inhomogeneous equation.

Solutions to the Problems

Lecture 12 | The Wronskian

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Suppose that we have determined that $x = X_1(t)$ and $x = X_2(t)$ are solutions to

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0,$$

and that we now attempt to write the general solution to the ode as

$$x = c_1 X_1(t) + c_2 X_2(t).$$

We must then ask whether this general solution can satisfy the two initial conditions given by

$$x(t_0) = x_0, \quad \dot{x}(t_0) = u_0.$$

Applying these initial conditions to our proposed general solution, we obtain

$$c_1 X_1(t_0) + c_2 X_2(t_0) = x_0, \quad c_1 \dot{X}_1(t_0) + c_2 \dot{X}_2(t_0) = u_0,$$

which is a system of two linear equations for c_1 and c_2. A unique solution exists provided

$$W = \begin{vmatrix} X_1(t_0) & X_2(t_0) \\ \dot{X}_1(t_0) & \dot{X}_2(t_0) \end{vmatrix} = X_1(t_0)\dot{X}_2(t_0) - \dot{X}_1(t_0)X_2(t_0) \neq 0,$$

where the determinant given by W is called the Wronskian.

For example, with $\omega \neq 0$, the two solutions $X_1(t) = \cos\omega t$ and $X_2(t) = \sin\omega t$ have a nonzero Wronskian for all t since

$$W = (\cos\omega t)\,(\omega\cos\omega t) - (-\omega\sin\omega t)\,(\sin\omega t) = \omega.$$

When the Wronskian is not equal to zero, we say that the two solutions $X_1(t)$ and $X_2(t)$ are linearly independent. The concept of linear independence is borrowed from linear algebra, and indeed, the set of all functions that satisfy a second-order linear homogeneous ode can be shown to form a two-dimensional vector space.

Problems for Lecture 12

1. Show that $X_1(t) = \exp(\alpha t)$ and $X_2(t) = \exp(\beta t)$ have a nonzero Wronskian for all t provided $\alpha \neq \beta$.

Solutions to the Problems

Lecture 13 | Homogeneous second-order ode with constant coefficients

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We now study solutions of the homogeneous constant-coefficient ode, written as

$$a\ddot{x} + b\dot{x} + cx = 0,$$

with a, b, and c constants. Our solution method finds two linearly independent solutions, multiplies each of these solutions by a constant, and adds them. The two free constants are then determined from initial values on x and $\dot{x}$.

Because of the differential properties of the exponential function, a natural ansatz, or educated guess, for the form of the solution is $x = e^{rt}$, where r is a constant to be determined. Successive differentiation results in $\dot{x} = re^{rt}$ and $\ddot{x} = r^2e^{rt}$, and substitution into the ode yields

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0.$$

Our choice of exponential function is now rewarded by the explicit cancelation of e^{rt}. The result is a quadratic equation for the unknown constant r:

$$ar^2 + br + c = 0.$$

Our ansatz has thus converted a differential equation for x into an algebraic equation for r. This algebraic equation is called the *characteristic equation* of the ode. Using the quadratic formula, the two solutions of the characteristic equation are given by

$$r_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider: (1) if $b^2 - 4ac > 0$, the roots are distinct and real ; (2) if $b^2 - 4ac < 0$, the roots are distinct and complex-conjugates; and (3) if $b^2 - 4ac = 0$, the roots are repeated. We will consider these three cases in turn.

Problems for Lecture 13

1. For the following differential equations, determine the roots of the characteristic equation.

a) $\ddot{x} - x = 0$;

b) $\ddot{x} + x = 0$;

c) $\ddot{x} - 2\dot{x} + x = 0$.

Solutions to the Problems

Practice Quiz | Theory of ode

1. Which two functions have a nonzero Wronskian?

a) $2e^{2t}, 3e^{2t}$

b) e^t, e^{t-t_0}

c) $\sin t, \sin{(t - \pi/2)}$

d) $\sin t, \sin{(t - \pi)}$

2. Let $x = X_1(t)$ and $x = X_2(t)$ be solutions to a homogeneous linear second-order ode. Which of the following functions is generally not a solution?

a) $x = \frac{1}{2}(X_1 + X_2)$

b) $x = \frac{1}{2}(X_1 - X_2)$

c) $x = 0$

d) $x = X_1(t)X_2(t)$

3. Which of the following odes has a characteristic equation with complex-conjugate roots?

I. $\ddot{x} + \dot{x} + x = 0$
II. $\ddot{x} + \dot{x} - x = 0$
III. $\ddot{x} - \dot{x} + x = 0$
IV. $\ddot{x} - \dot{x} - x = 0$

a) I only

b) IV only

c) I and III

d) II and IV

Solutions to the Practice quiz

Lecture 14 | Case 1: Distinct real roots

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When the roots of the characteristic equation are distinct and real, then the general solution to the second-order ode can be written as a linear superposition of the two solutions $e^{r_1 t}$ and $e^{r_2 t}$; that is,

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

The unknown constants c_1 and c_2 can then be determined by the given initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = u_0$.

Example: Solve $\ddot{x} + 5\dot{x} + 6x = 0$ *with* $x(0) = 2$, $\dot{x}(0) = 3$.

We take as our ansatz $x = e^{rt}$ and obtain the characteristic equation

$$r^2 + 5r + 6 = 0,$$

which factors to

$$(r+3)(r+2) = 0.$$

The general solution to the ode is thus

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

The solution for $\dot{x}$ obtained by differentiation is

$$\dot{x}(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}.$$

Use of the initial conditions then results in two equations for the two unknown constant c_1 and c_2,

$$c_1 + c_2 = 2, \quad -2c_1 - 3c_2 = 3,$$

with solution $c_1 = 9$ and $c_2 = -7$. Therefore, the unique solution that satisfies both the ode and the initial conditions is

$$\begin{aligned} x(t) &= 9e^{-2t} - 7e^{-3t} \\ &= 9e^{-2t}\left(1 - \frac{7}{9}e^{-t}\right). \end{aligned}$$

Problems for Lecture 14

1. Solve the following initial value problem: $\ddot{x} + 4\dot{x} + 3x = 0$, with $x(0) = 1$ and $\dot{x}(0) = 0$.

2. Find the solution of $\ddot{x} - x = 0$, with $x(0) = x_0$ and $\dot{x}(0) = u_0$.

Solutions to the Problems

Lecture 15 | Case 2: Complex-conjugate roots (Part A)

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When the roots of the characteristic equation are complex conjugates, we can define real numbers λ and μ such that the two roots are given by

$$r = \lambda + i\mu, \quad \bar{r} = \lambda - i\mu.$$

We have thus found the following two complex exponential solutions to the differential equation:

$$z(t) = e^{\lambda t} e^{i\mu t}, \quad \bar{z}(t) = e^{\lambda t} e^{-i\mu t}.$$

Applying the principle of superposition, any linear combination of z and $\bar{z}$ is also a solution to the second-order ode, and we can form two different linear combinations of $z(t)$ and $\bar{z}(t)$ that are real, namely $x_1(t) = \operatorname{Re} z(t)$ and $x_2(t) = \operatorname{Im} z(t)$. We have

$$x_1(t) = e^{\lambda t} \cos \mu t, \quad x_2(t) = e^{\lambda t} \sin \mu t.$$

Having found these two real solutions, we can then apply the principle of superposition a second time to determine the general solution for $x(t)$:

$$x(t) = e^{\lambda t} \left(A \cos \mu t + B \sin \mu t \right).$$

The real part of the roots of the characteristic equation appear in the exponential function, the imaginary part appears in the cosine and sine.

Lecture 16 | Case 2: Complex-conjugate roots (Part B)

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Example: Solve $\ddot{x} + \dot{x} + x = 0$ with $x(0) = 1$ and $\dot{x}(0) = 0$.

The characteristic equation is $r^2 + r + 1 = 0$, with roots

$$r = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \bar{r} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The general solution of the ode is therefore

$$x(t) = e^{-t/2}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right).$$

The derivative is

$$\dot{x}(t) = -\frac{1}{2}e^{-t/2}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}e^{-t/2}\left(-A\sin\frac{\sqrt{3}}{2}t + B\cos\frac{\sqrt{3}}{2}t\right).$$

Applying the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ results in

$$A = 1, \quad -\frac{1}{2}A + \frac{\sqrt{3}}{2}B = 0,$$

with solution $A = 1$ and $B = \sqrt{3}/3$. Therefore, the resulting solution is

$$x(t) = e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right).$$

Problems for Lecture 16

1. Solve $\ddot{x} - 2\dot{x} + 5x = 0$, with $x(0) = 1$ and $\dot{x}(0) = 0$.

2. Solve $\ddot{x} + x = 0$, with $x(0) = x_0$ and $\dot{x}(0) = u_0$.

Solutions to the Problems

Lecture 17 | Case 3: Repeated roots (Part A)

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If the exponential ansatz yields only one linearly independent solution, we need to find the missing second solution. We will do this by starting with the roots of the characteristic equation given by $r = \lambda \pm i\mu$, and taking the limit as μ goes to zero.

Now, the general solution for the case of complex-conjugate roots is given by

$$x(t) = e^{\lambda t}\left(A\cos\mu t + B\sin\mu t\right),$$

and to limit this solution as $\mu \to 0$ requires first satisfying the general initial conditions $x(0) = x_0$ and $\dot{x}(0) = u_0$. Solving for A and B, the solution becomes

$$x(t;\mu) = e^{\lambda t}\left(x_0\cos\mu t + \frac{u_0 - \lambda x_0}{\mu}\sin\mu t\right),$$

where we have written $x = x(t;\mu)$ to show explicitly that x now depends on the parameter μ.

Taking the limit as $\mu \to 0$, and using $\lim_{\mu\to 0}\mu^{-1}\sin\mu t = t$, we have

$$\lim_{\mu\to 0} x(t;\mu) = e^{\lambda t}\left(x_0 + (u_0 - \lambda x_0)t\right);$$

and the second solution is observed to be a constant, $u_0 - \lambda x_0$, times t times the first solution. The general solution for the case of repeated roots can therefore be written in the form

$$x(t) = (c_1 + c_2 t)e^{rt},$$

where r is the repeated root. The main result is that the second solution is t times the first solution.

Lecture 18 | Case 3: Repeated roots (Part B)

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Example: Solve $\ddot{x} + 2\dot{x} + x = 0$ *with* $x(0) = 1$ *and* $\dot{x}(0) = 0$.

The characteristic equation is $r^2 + 2r + 1 = (r+1)^2 = 0$, which has a repeated root given by $r = -1$. Therefore, the general solution to the ode is

$$x(t) = (c_1 + c_2 t)e^{-t}, \quad \dot{x}(t) = (c_2 - c_1 - c_2 t)e^{-t}.$$

Applying the initial conditions, we have $c_1 = 1$ and $c_2 - c_1 = 0$, so that $c_1 = c_2 = 1$. Therefore, the solution is

$$x(t) = (1 + t)e^{-t}.$$

Problems for Lecture 18

1. Solve $\ddot{x} - 2\dot{x} + x = 0$, with $x(0) = 1$ and $\dot{x}(0) = 0$.

Solutions to the Problems

Practice Quiz | Homogeneous equations

1. The solution of $\ddot{x} - 3\dot{x} + 2x = 0$ with initial values $x(0) = 1$ and $\dot{x}(0) = 0$ is given by

a) $e^{2t}(2 - e^{-t})$

b) $-e^{2t}(1 - 2e^{-t})$

c) $\frac{1}{2}e^{3t}(3 - e^{-2t})$

d) $-\frac{1}{2}e^{3t}(1 - 3e^{-2t})$

2. The solution of $\ddot{x} - 2\dot{x} + 2x = 0$ with initial values $x(0) = 1$ and $\dot{x}(0) = 0$ is given by

a) $e^{t}(\cos t + \sin t)$

b) $e^{t}(\cos t - \sin t)$

c) $e^{t}(\cos 2t - \frac{1}{2}\sin 2t)$

d) $e^{2t}(\cos t - 2\sin 2t)$

3. The solution of $\ddot{x} + 2\dot{x} + x = 0$ with initial values $x(0) = 0$ and $\dot{x}(0) = 1$ is given by

a) te^{-t}

b) $e^{-t}(1 - t)$

c) te^{t}

d) $e^{t}(1 - t)$

Solutions to the Practice quiz

Week III

Inhomogeneous Linear Differential Equations

We now add an inhomogeneous term to the second-order ode with constant coefficients. The inhomogeneous term may be an exponential, a sine or cosine, or a polynomial. We further study the phenomena of resonance, when the forcing frequency is equal to the natural frequency of the oscillator. Finally, we learn about three important applications: the RLC electrical circuit, a mass on a spring, and the pendulum.

Lecture 19 | Inhomogeneous second-order ode

View this lecture on YouTube

The inhomogeneous linear second-order ode is given by

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t),$$

with initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = u_0$. There is a three-step solution method when the inhomogeneous term $g(t) \neq 0$. (1) Solve the homogeneous ode

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0,$$

for two independent solutions $x = x_1(t)$ and $x = x_2(t)$, and form a linear superposition to obtain the general solution

$$x_h(t) = c_1 x_1(t) + c_2 x_2(t),$$

where c_1 and c_2 are free constants. (2) Find a *particular* solution $x = x_p(t)$ that solves the inhomogeneous ode. A particular solution is most easily found when $p(t)$ and $q(t)$ are constants, and when $g(t)$ is a combination of polynomials, exponentials, sines and cosines. (3) Write the general solution of the inhomogeneous ode as the sum of the homogeneous and particular solutions,

$$x(t) = x_h(t) + x_p(t),$$

and apply the initial conditions to determine the constants c_1 and c_2 appearing in the homogeneous solution. Note that because of the linearity of the differential equations,

$$\begin{aligned}
\ddot{x} + p\dot{x} + qx &= \frac{d^2}{dt^2}(x_h + x_p) + p\frac{d}{dt}(x_h + x_p) + q(x_h + x_p) \\
&= (\ddot{x}_h + p\dot{x}_h + qx_h) + (\ddot{x}_p + p\dot{x}_p + qx_p) \\
&= 0 + g \\
&= g,
\end{aligned}$$

so that the sum of the homogeneous and particular solutions solve the ode, and the two free constants in x_h can be used to satisfy the two initial conditions.

Problems for Lecture 19

1. Consider the inhomogeneous linear second-order ode given by

$$\ddot{x} + p(t)\dot{x} + q(t)x = g_1(t) + g_2(t).$$

Show that

$$x(t) = x_h(t) + x_{p_1}(t) + x_{p_2}(t)$$

is the general solution, where $x_h(t)$ is the general solution to the homogeneous ode, $x_{p_1}(t)$ is a particular solution for the inhomogeneous ode with only $g_1(t)$ on the right-hand-side, and $x_{p_2}(t)$ is a particular solution for the inhomogeneous ode with only $g_2(t)$ on the right-hand side.

Solutions to the Problems

Lecture 20 | Inhomogeneous term: Exponential function

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Example: Solve $\ddot{x} - 3\dot{x} - 4x = 3e^{2t}$ *with* $x(0) = 1$ *and* $\dot{x}(0) = 0$.

First, we solve the homogeneous equation. The characteristic equation is

$$r^2 - 3r - 4 = (r-4)(r+1) = 0,$$

so that

$$x_h(t) = c_1 e^{4t} + c_2 e^{-t}.$$

Second, we find a particular solution of the inhomogeneous equation. We try an ansatz such that the exponential function cancels:

$$x(t) = Ae^{2t},$$

where A is an undetermined coefficient. Substituting our ansatz into the ode, we obtain

$$4A - 6A - 4A = 3,$$

from which we determine $A = -1/2$. Obtaining a solution for A independent of t justifies the ansatz. Third, we write the general solution to the ode as the sum of the homogeneous and particular solutions, and determine c_1 and c_2 that satisfy the initial conditions. We have

$$x(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t}, \qquad \dot{x}(t) = 4c_1 e^{4t} - c_2 e^{-t} - e^{2t}.$$

Applying the initial conditions yields the two linear equations

$$c_1 + c_2 = \frac{3}{2}, \qquad 4c_1 - c_2 = 1,$$

with solution $c_1 = 1/2$ and $c_2 = 1$. Therefore, the solution for $x(t)$ that satisfies both the ode and the initial conditions is given by

$$x(t) = \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} + e^{-t} = \frac{1}{2}e^{4t}\left(1 - e^{-2t} + 2e^{-5t}\right),$$

where we have grouped the terms in the final solution to better display the asymptotic behavior for large t.

Problems for Lecture 20

1. Solve $\ddot{x} + 5\dot{x} + 6x = e^{-t}$, with $x(0) = 0$ and $\dot{x}(0) = 0$.

Solutions to the Problems

Practice Quiz | Solving inhomogeneous equations

All the questions will consider the differential equation given by

$$\ddot{x} + 5\dot{x} + 6x = 2e^{-t}.$$

1. What is the solution that satisfies $x(0) = 0$ and $\dot{x}(0) = 0$?

a) $x(t) = e^{-t}(1 - 2e^{-t} + e^{-2t})$

b) $x(t) = e^{t}(1 - 4e^{-3t} + 3e^{-4t})$

c) $x(t) = e^{3t}(3 - 4e^{-t} + e^{-4t})$

d) $x(t) = e^{3t}(1 - 2e^{-t} + e^{-2t})$

2. What is the solution that satisfies $x(0) = 1$ and $\dot{x}(0) = 0$?

a) $x(t) = e^{-t}(1 + e^{-t} - e^{-2t})$

b) $x(t) = e^{t}(1 - e^{-3t} + e^{-4t})$

c) $x(t) = e^{3t}(1 - e^{-t} + e^{-4t})$

d) $x(t) = -e^{3t}(1 - e^{-t} - e^{-2t})$

3. What is the solution that satisfies $x(0) = 0$ and $\dot{x}(0) = 1$?

a) $x(t) = e^{-t}(1 - e^{-t})$

b) $x(t) = e^{t}(1 - 3e^{-3t} + 2e^{-4t})$

c) $x(t) = e^{3t}(4 - 5e^{-t} + e^{-4t})$

d) $x(t) = e^{3t}(2 - 3e^{-t} + e^{-2t})$

Solutions to the Practice quiz

Lecture 21 | Inhomogeneous term: Sine or cosine (Part A)

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Example: Find a particular solution of $\ddot{x} - 3\dot{x} - 4x = 2\sin t$.

We will demonstrate two solution methods in this and the next lecture. The first method tries the ansatz

$$x(t) = A\cos t + B\sin t,$$

where the argument of cosine and sine must agree with the argument of sine in the inhomogeneous term. The cosine term is required because the derivative of sine is cosine. Upon substitution into the differential equation, we obtain

$$(-A\cos t - B\sin t) - 3\left(-A\sin t + B\cos t\right) - 4\left(A\cos t + B\sin t\right) = 2\sin t,$$

or regrouping terms,

$$-\left(5A + 3B\right)\cos t + \left(3A - 5B\right)\sin t = 2\sin t.$$

This equation is valid for all t, and in particular for $t = 0$ and $\pi/2$, for which the sine and cosine functions vanish. For these two values of t, we find

$$5A + 3B = 0, \qquad 3A - 5B = 2;$$

and solving we obtain $A = 3/17$ and $B = -5/17$. The particular solution is therefore given by

$$x_p = \frac{1}{17}\left(3\cos t - 5\sin t\right).$$

Lecture 22 | Inhomogeneous term: Sine or cosine (Part B)

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Example: Find a particular solution of $\ddot{x} - 3\dot{x} - 4x = 2\sin t$.

The second method makes use of the relation $e^{it} = \cos t + i \sin t$ to convert the sine inhomogeneous term to an exponential function. We introduce the complex function $z = z(t)$ and note that $\sin t = \operatorname{Im}\{e^{it}\}$. We then consider the complex ode

$$\ddot{z} - 3\dot{z} - 4z = 2e^{it},$$

where $x = \operatorname{Im}\{z\}$ satisfies the original differential equation for x.

To find a particular solution of the complex equation, we try the ansatz $z(t) = Ce^{it}$, where we now expect C to be a complex constant. Upon substitution into the complex ode, and using $i^2 = -1$, we obtain $-C - 3iC - 4C = 2$; or upon solving for C,

$$C = \frac{-2}{5+3i} = \frac{-5+3i}{17}.$$

Therefore,

$$x_p = \operatorname{Im}\{z_p\} = \operatorname{Im}\left\{\frac{1}{17}(-5+3i)(\cos t + i \sin t)\right\} = \frac{1}{17}(3\cos t - 5\sin t),$$

the same result as in the first method.

Problems for Lecture 22

1. Find a particular solution of $\ddot{x} - 3\dot{x} - 4x = 2\cos t$ by trying a

a) cosine and sine ansatz;

b) an exponential ansatz.

Solutions to the Problems

Lecture 23 | Inhomogeneous term: Polynomials

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Example: Find a particular solution of $\ddot{x} + \dot{x} - 2x = t^2$.
Our ansatz should be a polynomial in t of the same order as the inhomogeneous term. Accordingly, we try

$$x(t) = At^2 + Bt + C.$$

Upon substitution into the ode, we have

$$2A + (2At + B) - 2(At^2 + Bt + C) = t^2,$$

or

$$-2At^2 + 2(A - B)t + (2A + B - 2C)t^0 = t^2.$$

Equating powers of t,

$$-2A = 1, \quad 2(A - B) = 0, \quad 2A + B - 2C = 0;$$

and solving,

$$A = -\frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{3}{4}.$$

The particular solution is therefore

$$x_p(t) = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}.$$

Problems for Lecture 23

1. Find a particular solution of $\ddot{x} + \dot{x} + x = t$.

Solutions to the Problems

Practice Quiz | Particular solutions

1. A particular solution of $\ddot{x} + 3\dot{x} + 2x = 2e^{2t}$ is given by

a) $x_p(t) = \dfrac{1}{6}e^{-2t}$

b) $x_p(t) = \dfrac{1}{6}e^{2t}$

c) $x_p(t) = -\dfrac{1}{6}e^{-2t}$

d) $x_p(t) = -\dfrac{1}{6}e^{2t}$

2. A particular solution of $\ddot{x} - \dot{x} - 2x = 2\cos 2t$ is given by

a) $x_p(t) = \dfrac{1}{10}(\cos 2t + 3\sin 2t)$

b) $x_p(t) = -\dfrac{1}{10}(\cos 2t + 3\sin 2t)$

c) $x_p(t) = \dfrac{1}{10}(3\cos 2t + \sin 2t)$

d) $x_p(t) = -\dfrac{1}{10}(3\cos 2t + \sin 2t)$

3. A particular solution of $\ddot{x} - 3\dot{x} + 2x = t + 1$ is given by

a) $x_p(t) = \dfrac{5}{4}t + \dfrac{1}{2}$

b) $x_p(t) = \dfrac{5}{4}t - \dfrac{1}{2}$

c) $x_p(t) = \dfrac{1}{2}t + \dfrac{5}{4}$

d) $x_p(t) = \dfrac{1}{2}t - \dfrac{5}{4}$

Solutions to the Practice quiz

Lecture 24 | Resonance

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Resonance occurs when the forcing frequency ω matches the natural frequency ω_0 of an oscillator. To illustrate resonance, we consider the inhomogeneous linear second-order ode

$$\ddot{x} + \omega_0^2 x = f\cos\omega t, \quad x(0) = 0,\ \dot{x}(0) = 0;$$

and determine what happens to the solution in the limit $\omega \to \omega_0$.

The homogeneous equation has characteristic equation $r^2 + \omega_0^2 = 0$, with solution $r = \pm i\omega_0$, so that the solution to the homogeneous equation is

$$x_h(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

A particular solution of the equation can be found by trying $x(t) = A\cos\omega t$. Upon substitution into the ode, we can solve for A to obtain $A = f/(\omega_0^2 - \omega^2)$, so that

$$x_p(t) = \frac{f}{\omega_0^2 - \omega^2}\cos\omega t.$$

Our general solution is thus

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{f}{\omega_0^2 - \omega^2}\cos\omega t.$$

Initial conditions are satisfied when $c_1 = -f/(\omega_0^2 - \omega^2)$ and $c_2 = 0$, so that

$$x(t) = \frac{f(\cos\omega t - \cos\omega_0 t)}{\omega_0^2 - \omega^2}.$$

Resonance occurs in the limit $\omega \to \omega_0$; that is, the frequency of the inhomogeneous term (the external force) matches the frequency of the homogeneous solution (the free oscillation). By L'Hospital's rule, the limit of $x = x(t)$ is found by differentiating with respect to ω:

$$\lim_{\omega\to\omega_0} x(t) = \lim_{\omega\to\omega_0}\frac{f(\cos\omega t - \cos\omega_0 t)}{\omega_0^2 - \omega^2} = \lim_{\omega\to\omega_0}\frac{-ft\sin\omega t}{-2\omega} = \frac{ft\sin\omega_0 t}{2\omega_0}.$$

At resonance, the oscillation increases linearly with t, resulting in a large amplitude oscillation.

More generally, if the inhomogeneous term in the differential equation is a solution of the corresponding homogeneous differential equation, then one should multiply the usual ansatz for the particular solution by t.

Problems for Lecture 24

1. Solve $\ddot{x} + 3\dot{x} + 2x = e^{-2t}$, with $x(0) = 0$ and $\dot{x}(0) = 0$. Observe that the inhomogeneous term is a solution of the homogeneous equation. To find a particular solution, the usual exponential ansatz must be multiplied by t.

Solutions to the Problems

Lecture 25 | Application: RLC circuit

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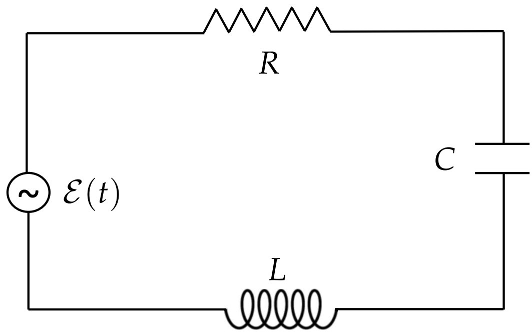

Consider a resister R, an inductor L and a capacitor C connected in series as shown in the above figure. An AC generator provides a time-varying electromotive force to the circuit, given by $\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t$. The equations for the voltage drops across a capacitor, a resister and an inductor are

$$V_C = q/C, \quad V_R = iR, \quad V_L = L\,di/dt,$$

where C is the capacitance, R is the resistance and L is the inductance. The charge q and the current i are related by $i = dq/dt$.

Kirchhoff's voltage law states that the electromotive force applied to any closed loop is equal to the sum of the voltage drops in that loop. Appying Kirchhoff's law, we have $V_L + V_R + V_C = \mathcal{E}(t)$, or

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \mathcal{E}_0 \cos \omega t. \tag{25.1}$$

This is a second-order linear inhomogeneous differential equation with constant coefficients.

To reduce the number of free parameters in this equation, we can nondimensionalize. We first define the natural frequency of oscillation of a system to be the frequency of oscillation in the absence of any driving or damping forces. For the RLC circuit, the natural frequency of oscillation is given by $\omega_0 = 1/\sqrt{LC}$, and making use of ω_0, we can define a dimensionless time τ and a dimensionless charge Q by

$$\tau = \omega_0 t, \quad Q = \frac{\omega_0^2 L}{\mathcal{E}_0} q.$$

The resulting dimensionless equation for the RLC circuit can then be found to be

$$\frac{d^2Q}{d\tau^2} + \alpha\frac{dQ}{d\tau} + Q = \cos \beta\tau, \tag{25.2}$$

where α and β are dimensionless parameters given by

$$\alpha = \frac{R}{L\omega_0}, \quad \beta = \frac{\omega}{\omega_0}.$$

Notice that the original five parameters in (25.1), namely R, L, C, $\mathcal{E}_0$ and ω, have been reduced to the two dimensionless parameters α and β in (25.2).

Problems for Lecture 25

1. Determine how to nondimensionalize the RLC circuit equation so that the dimensionless equation takes the form

$$\alpha \frac{d^2 Q}{d\tau^2} + \frac{dQ}{d\tau} + Q = \cos \beta\tau.$$

What are the definitions of τ, Q, α and β?

Solutions to the Problems

Lecture 26 | Application: Mass on a spring

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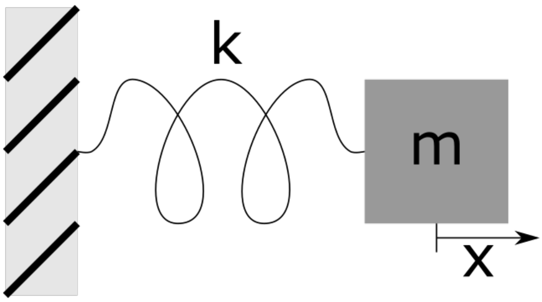

Consider a mass lying on a flat surface and connected by a spring to a wall, as shown above. The spring force is modeled by Hooke's law, $F_s = -kx$, and sliding friction is modeled as $F_f = -c\,dx/dt$. An external force is applied and is assumed to be sinusoidal, with $F_e = F_0 \cos \omega t$. Newton's equation, $F = ma$, results in

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \cos \omega t,$$

another second-order linear inhomogeneous differential equation with constant coefficients. Here, the natural frequency of oscillation is given by $\omega_0 = \sqrt{k/m}$, and we define a dimensionless time τ and a dimensionless position X by

$$\tau = \omega_0 t, \quad X = \frac{m\omega_0^2}{F_0}x.$$

The resulting dimensionless equation is given by

$$\frac{d^2X}{d\tau^2} + \alpha\frac{dX}{d\tau} + X = \cos \beta\tau,$$

where here, α and β are dimensionless parameters given by

$$\alpha = \frac{c}{m\omega_0}, \quad \beta = \frac{\omega}{\omega_0}.$$

Even though the physical problem is different, the resulting dimensionless equation for the mass-spring system is the same as that for the RLC circuit.

Problems for Lecture 26

1. Determine how to nondimensionalize the mass on a spring equation so that the dimensionless equation takes the form

$$\alpha \frac{d^2X}{d\tau^2} + \frac{dX}{d\tau} + X = \cos \beta\tau.$$

What are the definitions of τ, X, α and β?

Solutions to the Problems

Lecture 27 | Application: Pendulum

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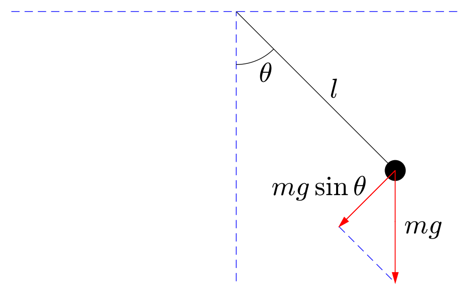

Consider a mass attached to a massless rigid rod and constrained to move along an arc of a circle centered at the pivot point (see figure). Suppose l is the fixed length of the connecting rod, and θ is the angle it makes with the vertical.

We apply Newton's equation, $F = ma$, to the mass with origin at the bottom and axis along the arc with positive direction to the right. The position s of the mass along the arc is given by $s = l\theta$. The relevant gravitational force on the pendulum is the component along the arc, and is given by $F_g = -mg\sin\theta$. We model friction to be proportional to the velocity of the pendulum along the arc, that is $F_f = -c\dot{s} = -cl\dot{\theta}$. With a sinusoidal external force, $F_e = F_0\cos\omega t$, Newton's equation $m\ddot{s} = F_g + F_f + F_e$ results in

$$ml\ddot{\theta} + cl\dot{\theta} + mg\sin\theta = F_0\cos\omega t.$$

At small amplitudes of oscillation, we can approximate $\sin\theta \approx \theta$, and the natural frequency of oscillation ω_0 of the mass is given by $\omega_0 = \sqrt{g/l}$. Nondimensionalizing time as $\tau = \omega_0 t$, the dimensionless pendulum equation becomes

$$\frac{d^2\theta}{d\tau^2} + \alpha\frac{d\theta}{d\tau} + \sin\theta = \gamma\cos\beta\tau,$$

where α, β, and γ are dimensionless parameters given by

$$\alpha = \frac{c}{m\omega_0}, \quad \beta = \frac{\omega}{\omega_0}, \quad \gamma = \frac{F_0}{ml\omega_0^2}.$$

The nonlinearity of the pendulum equation, with the term $\sin\theta$, results in the additional dimensionless parameter γ. For small amplitude of oscillation, however, we can scale θ by $\theta = \gamma\Theta$, and the small amplitude dimensionless equation becomes

$$\frac{d^2\Theta}{d\tau^2} + \alpha\frac{d\Theta}{d\tau} + \Theta = \cos\beta\tau,$$

the same equation as the dimensionless equations for the RLC circuit and the mass on a spring.

Lecture 28 | Damped resonance

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The dimensionless odes for the RLC circuit, mass on a spring, and small amplitude pendulum have the form

$$\ddot{x} + \alpha\dot{x} + x = \cos\beta t, \tag{28.1}$$

where the physical constraints of these applications require that $\alpha > 0$. The corresponding homogeneous equation has the characteristic polynomial $r^2 + \alpha r + 1 = 0$, with roots $r_\pm = (-\alpha \pm \sqrt{\alpha^2 - 4})/2$. No matter the sign of the discriminant, we have $\text{Re}\{r_\pm\} < 0$. Therefore, both linearly independent homogeneous solutions decay exponentially to zero, and the long-time solution of the differential equation reduces to the non-decaying particular solution. Since the initial conditions are satisfied by the free constants multiplying the decaying homogeneous solutions, the long-time solution is independent of the initial conditions.

If we are only interested in the long-time solution of (28.1), we only need determine the particular solution. We can consider the complex ode

$$\ddot{z} + \alpha\dot{z} + z = e^{i\beta t},$$

with $x_p = \text{Re}(z_p)$. With the ansatz $z_p = Ae^{i\beta t}$, we have $-\beta^2 A + i\alpha\beta A + A = 1$, or

$$A = \frac{1}{(1-\beta^2) + i\alpha\beta}.$$

Of particular interest is when the forcing frequency ω is equal to the natural frequency ω_0 of the undampled oscillator. In the dimensionless equation of (28.1), this resonance situation corresponds to $\beta = 1$. In this case, $A = 1/i\alpha$ and

$$x_p = \text{Re}\{e^{it}/i\alpha\} = (1/\alpha)\sin t.$$

The oscillator position is observed to be $\pi/2$ out of phase with the external force, or in other words, the velocity of the oscillator, not the position, is in phase with the force. Also, the amplitude of oscillation is seen to be inversely proportional to the damping coefficient represented by α. Smaller and smaller damping will obviously lead to larger and larger oscillations.

Problems for Lecture 28

1. Consider the differential equation

$$\ddot{x} + \alpha\dot{x} + x = \cos\beta t.$$

Find the long-time amplitude of oscillation as a function of α and β.

Solutions to the Problems

Practice Quiz | Applications and resonance

1. The solution of the differential equation $\ddot{x} + \dot{x} = 1$, with $x(0) = 0$ and $\dot{x}(0) = 0$, is given by

a) $t(1 - e^{t})$

b) $t(1 - e^{-t})$

c) $(t + 1) - e^{t}$

d) $(t - 1) + e^{-t}$

2. The solution of the differential equation $\ddot{x} - x = \cosh t$, with $x(0) = 0$ and $\dot{x}(0) = 0$, is given by

a) $\frac{1}{2} t \cosh t$

b) $\frac{1}{2} t \sinh t$

c) $\frac{1}{2}(t \cosh t + \sinh t)$

d) $\frac{1}{2}(t \cosh t - \sinh t)$

3. When comparing the RLC circuit to the mass on a spring and to the pendulum, the resistor R plays the role of

a) mass

b) gravity

c) friction

d) restoring force

Solutions to the Practice quiz

Week IV

Laplace Transform and Series Solution Methods

We present here two new analytical solution methods for solving linear differential equations. The first is the Laplace transform method, which is used to solve the constant-coefficient ode with a discontinuous or impulsive inhomogeneous term. The Laplace transform is a good vehicle in general for introducing sophisticated integral transform techniques within an easily understandable context. We also discuss the series solution of a linear differential equation. Although we do not go deeply here, an introduction to this technique may be useful to students that encounter it again in more advanced courses.

Lecture 29 | Definition of the Laplace transform

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The Laplace transform of $f(t)$, denoted by $F(s) = \mathcal{L}\{f(t)\}$, is defined by the integral

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

The values of s may be restricted to ensure convergence. The Laplace transform can be shown to be a linear transformation. We have

$$\begin{aligned}\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}\big(c_1 f_1(t) + c_2 f_2(t)\big)\, dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt \\ &= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.\end{aligned}$$

There is also a one-to-one correspondence between functions and their Laplace transforms, and a table of Laplace transforms is used to find both Laplace and inverse Laplace transforms of commonly occurring functions.

To construct such a table, integrals have been performed, where for example, we can compute

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-(s-a)t}\, dt \\ &= -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^\infty \\ &= \frac{1}{s-a}.\end{aligned}$$

Here, the assumption is made that $s > a$.

A Table of Laplace transforms can be found in Appendix F.

Problems for Lecture 29

1. Compute the Laplace transform of $f(t) = \sin bt$.

Solutions to the Problems

Lecture 30 | Laplace transform of a constant-coefficient ode

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Consider the inhomogeneous constant-coefficient second-order ode, given by

$$a\ddot{x} + b\dot{x} + cx = g(t), \quad x(0) = x_0, \;\; \dot{x}(0) = u_0.$$

We Laplace transform the ode making use of the linearity of the transform:

$$a\mathcal{L}\{\ddot{x}\} + b\mathcal{L}\{\dot{x}\} + c\mathcal{L}\{x\} = \mathcal{L}\{g\}.$$

We define $X(s) = \mathcal{L}\{x(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. The Laplace transforms of the derivatives can be found from integrating by parts. We have

$$\int_0^\infty e^{-st}\dot{x}\,dt = xe^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}x\,dt = sX(s) - x_0;$$

and

$$\int_0^\infty e^{-st}\ddot{x}\,dt = \dot{x}e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}\dot{x}\,dt = -u_0 + s\big(sX(s) - x_0\big) = s^2X(s) - sx_0 - u_0.$$

The transformed ode is then

$$a\left(s^2X - sx_0 - u_0\right) + b\left(sX - x_0\right) + cX = G,$$

which is an easily solved linear algebraic equation for $X = X(s)$. Finding the solution of the ode then requires taking the inverse Laplace transform of $X = X(s)$ to obtain $x = x(t)$.

Problems for Lecture 30

1. Consider the inhomogeneous constant-coefficient second-order ode given by

$$a\ddot{x} + b\dot{x} + cx = g(t), \quad x(0) = x_0, \ \ \dot{x}(0) = u_0.$$

Determine the solution for $X = X(s)$ in terms of the Laplace transform of $g(t)$.

Solutions to the Problems

Lecture 31 | Solution of an initial value problem

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Example: Solve by Laplace transform methods $\ddot{x} + x = \sin 2t$ *with* $x(0) = 2$ *and* $\dot{x}(0) = 1$.

Taking the Laplace transform of both sides of the ode, we find

$$s^2 X(s) - 2s - 1 + X(s) = \frac{2}{s^2+4},$$

where the Laplace transform of the second derivative and of $\sin 2t$ made use of line 16 and line 6a of the table in Appendix F. Solving for $X(s)$, we obtain

$$X(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}.$$

To determine the inverse Laplace transform from the table in Appendix F, we need to perform a partial fraction expansion of the second term:

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}.$$

By inspection, we can observe that $a = c = 0$ and that $d = -b$. A quick calculation shows that $3b = 2$, or $b = 2/3$. Therefore,

$$\begin{aligned} X(s) &= \frac{2s+1}{s^2+1} + \frac{2/3}{s^2+1} - \frac{2/3}{(s^2+4)} \\ &= \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{(s^2+4)}. \end{aligned}$$

From lines 6a and 7a of the table in Appendix F, we obtain the solution by taking inverse Laplace transforms of the three terms separately, where the values in the table are $b = 1$ in the first two terms, and $b = 2$ in the third term:

$$x(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$

Problems for Lecture 31

1. Solve by Laplace transform methods $\ddot{x} + 5\dot{x} + 6x = e^{-t}$, with $x(0) = 0$ and $\dot{x}(0) = 0$.

Solutions to the Problems

Practice Quiz | The Laplace transform method

1. What is the Laplace transform of $x(t) = e^{-t}\cos \pi t$?

a) $\dfrac{s+1}{(s+1)^2 - \pi^2}$

b) $\dfrac{s+1}{(s+1)^2 + \pi^2}$

c) $\dfrac{s-1}{(s-1)^2 - \pi^2}$

d) $\dfrac{s-1}{(s-1)^2 + \pi^2}$

2. If $\ddot{x} + \dot{x} - 6x = e^{-t}$, with $x(0) = \dot{x}(0) = 0$, what is $X(s)$?

a) $\dfrac{1}{(s+1)(s+2)(s+3)}$

b) $\dfrac{1}{(s-1)(s+2)(s+3)}$

c) $\dfrac{1}{(s+1)(s-2)(s+3)}$

d) $\dfrac{1}{(s+1)(s+2)(s-3)}$

3. If $X(s) = \dfrac{1}{(s+1)(s+2)(s+3)}$, what is $x(t)$?

a) $\frac{1}{2}e^{-t} + e^{-2t} + \frac{1}{2}e^{-3t}$

b) $-\frac{1}{2}e^{-t} + e^{-2t} + \frac{1}{2}e^{-3t}$

c) $\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}$

d) $\frac{1}{2}e^{-t} + e^{-2t} - \frac{1}{2}e^{-3t}$

Solutions to the Practice quiz

Lecture 32 | The Heaviside step function

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The Heaviside or unit step function, denoted here by $u_c(t)$, is zero for $t < c$ and one for $t \geq c$:

$$u_c(t) = \begin{cases} 0, & t < c; \\ 1, & t \geq c. \end{cases}$$

The Heaviside function can be viewed as the step-up function. Using the Heaviside function both a step-down and a step-up, step-down function can also be defined. The Laplace transform of the Heaviside function is determined by integration:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) dt = \frac{e^{-cs}}{s}.$$

The Heaviside function can be used to represent a translation of a function $f(t)$ a distance c in the positive t direction. We have

$$u_c(t) f(t-c) = \begin{cases} 0, & t < c; \\ \text{f(t-c)}, & t \geq c. \end{cases}$$

The Laplace transform is

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c) dt = e^{-cs} F(s).$$

The translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by the exponential e^{-cs}.

Piecewise-defined inhomogeneous terms can be modeled using Heaviside functions. For example, consider the general case of a piecewise function defined on two intervals:

$$f(t) = \begin{cases} f_1(t), & \text{if } t < c; \\ f_2(t), & \text{if } t \geq c. \end{cases}$$

Using the Heaviside function u_c, the function $f(t)$ can be written in a single line as

$$f(t) = f_1(t) + \big(f_2(t) - f_1(t)\big) u_c(t).$$

Problems for Lecture 32

1. Use the step-up function $u_c(t)$ to construct a step-down and a step-up, step-down function.

2. Prove that $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$.

3. Consider the piecewise continuous function given by

$$f(t) = \begin{cases} t, & \text{if } t < 1; \\ 1, & \text{if } t \geq 1. \end{cases}$$

a) Express $f(t)$ in a single line using the Heaviside function.

b) Find $F = F(s)$.

Solutions to the Problems

Lecture 33 | The Dirac delta function

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The Dirac delta function, denoted as $\delta(t)$, is defined by requiring that for any function $f(t)$,

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0).$$

The usual view of the Dirac delta function is that it is zero everywhere except at $t = 0$, at which it is infinite in such a way that the integral is one. The Dirac delta function is technically not a function, but is what mathematicians call a distribution. Nevertheless, in most cases of practical interest, it can be treated like a function, where physical results are obtained following a final integration.

There are several common ways to represent the Dirac delta function as a limit of a well-defined function. For our purposes, the most useful representation of the shifted Dirac delta function, $\delta(t-c)$, makes use of the step-up, step-down function constructed from Heaviside functions:

$$\delta(t-c) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\left(u_{c-\epsilon}(t) - u_{c+\epsilon}(t)\right).$$

Before taking the limit, the well-defined step-up, step-down function is zero except over a small interval of width 2ϵ centered at $t = c$, over which it takes the large value $1/2\epsilon$. The integral of this function is one, independent of the value of ϵ.

The Laplace transform of the Dirac delta function is easily found by integration using the definition of the delta function. With $c > 0$,

$$\mathcal{L}\{\delta(t-c)\} = \int_0^{\infty} e^{-st}\delta(t-c)\,dt = e^{-cs}.$$

Problems for Lecture 33

Remember that the Dirac delta function is well-defined only after it is integrated over.

1. Prove that $\delta(ax) = \dfrac{1}{|a|}\delta(x)$.

2. Show that $u_c(x) = \displaystyle\int_{-\infty}^{x} \delta(x' - c)\, dx'$.

3. Show that $\delta(x - c) = \dfrac{d}{dx} u_c(x)$.

Solutions to the Problems

Lecture 34 | Solution of a discontinuous inhomogeneous term

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Example: Solve $\ddot{x} + 3\dot{x} + 2x = 1 - u_1(t)$*, with* $x(0) = \dot{x}(0) = 0$

Here, the inhomogeneous term is a step-down function, from one to zero. Taking the Laplace transform of the ode using the table in Appendix F, we have

$$s^2X(s) + 3sX(s) + 2X(s) = \frac{1}{s}\left(1 - e^{-s}\right),$$

with solution for $X = X(s)$ given by

$$X(s) = \frac{1 - e^{-s}}{s(s+1)(s+2)}.$$

Defining

$$F(s) = \frac{1}{s(s+1)(s+2)},$$

and using the table in Appendix F, the inverse Laplace transform of $X(s)$ can be written as

$$x(t) = f(t) - u_1(t)f(t-1),$$

where $f(t)$ is the inverse Laplace transform of $F(s)$. To determine $f(t)$, we need a partial fraction expansion of $F(s)$, so we write

$$\frac{1}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2}.$$

Using the coverup method, we find $a = 1/2$, $b = -1$, and $c = 1/2$, and from the table in Appendix F, determine

$$\begin{aligned}\mathcal{L}^{-1}\left\{F(s)\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.\end{aligned}$$

The full solution can then be written as

$$x(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u_1(t)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right).$$

Problems for Lecture 34

1. Show that the solution in the lecture,

$$x(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u_1(t)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right),$$

is continuous at $t = 1$.

2. Solve $\ddot{x} + x = 1 - u_{2\pi}(t)$, with $x(0) = 0$ and $\dot{x}(0) = 0$.

Solutions to the Problems

Lecture 35 | Solution of an impulsive inhomogeneous term

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Example: Solve $\ddot{x} + 3\dot{x} + 2x = \delta(t)$ *with* $x(0) = \dot{x}(0) = 0$*. Assume the entire impulse occurs at* $t = 0^+$*.*

Taking the Laplace transform of the ode using the table in Appendix F, and applying the initial conditions, we have

$$s^2 X + 3sX + 2X = 1.$$

Solving for $X = X(s)$, we have

$$X(s) = \frac{1}{(s+1)(s+2)},$$

and a partial fraction expansion results in

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse Laplace transform using the table in Appendix F, we find

$$\begin{aligned} x(t) &= e^{-t} - e^{-2t} \\ &= e^{-t}\left(1 - e^{-t}\right). \end{aligned}$$

Note that the function $x = x(t)$ is continuous at $t = 0$, but that the first derivative is not, since $\dot{x}(0^-) = 0$ and $\dot{x}(0^+) = 1$. Impulsive forces result in a discontinuity in the velocity.

Problems for Lecture 35

1. Solve $\ddot{x} + x = \delta(t) - \delta(t - 2\pi)$, with $x(0) = 0$ and $\dot{x}(0) = 0$.

Solutions to the Problems

Practice Quiz | Discontinuous and impulsive inhomogeneous terms

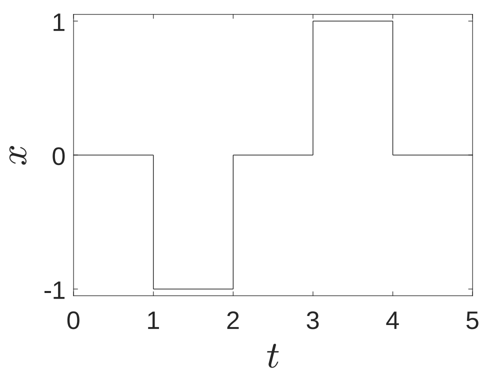
1. The pictured function can be defined using Heaviside step functions as

a) $x(t) = -u_1(t) + u_2(t) + u_3(t) - u_4(t)$

b) $x(t) = -u_1(t) + u_2(t) - u_3(t) + u_4(t)$

c) $x(t) = u_1(t) - u_2(t) - u_3(t) + u_4(t)$

d) $x(t) = u_1(t) - u_2(t) + u_3(t) - u_4(t)$



2. The solution to $\ddot{x} + x = 1 - u_{2\pi}(t)$, with $x(0) = 1$ and $\dot{x}(0) = 0$ is given by

a) $x(t) = \begin{cases} 1 - \cos t, & \text{if } t < 2\pi; \\ 0, & \text{if } t \geq 2\pi. \end{cases}$

b) $x(t) = \begin{cases} \cos t, & \text{if } t < 2\pi; \\ 1, & \text{if } t \geq 2\pi. \end{cases}$

c) $x(t) = \begin{cases} 1 - \sin t, & \text{if } t < 2\pi; \\ 1, & \text{if } t \geq 2\pi. \end{cases}$

d) $x(t) = \begin{cases} 1, & \text{if } t < 2\pi; \\ \cos t, & \text{if } t \geq 2\pi. \end{cases}$

3. The solution to $\ddot{x} + x = \delta(t) - \delta(t - 2\pi)$, with $x(0) = 1$ and $\dot{x}(0) = 0$ is given by

a) $x(t) = \begin{cases} \cos t, & \text{if } t < 2\pi; \\ 0, & \text{if } t \geq 2\pi. \end{cases}$

b) $x(t) = \begin{cases} \sin t, & \text{if } t < 2\pi; \\ 0, & \text{if } t \geq 2\pi. \end{cases}$

c) $x(t) = \begin{cases} \cos t + \sin t, & \text{if } t < 2\pi; \\ \cos t, & \text{if } t \geq 2\pi. \end{cases}$

d) $x(t) = \begin{cases} \cos t + \sin t, & \text{if } t < 2\pi; \\ \sin t, & \text{if } t \geq 2\pi. \end{cases}$

Solutions to the Practice quiz

Lecture 36 | The series solution method

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Example: Find the general solution of $y'' + y = 0$.

By now, you should know that the general solution is $y(x) = a_0 \cos x + a_1 \sin x$, with a_0 and a_1 constants. To find a power series solution, we write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n;$$

and upon differentiating term-by-term

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting the power series for y and its derivatives into the differential equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0. \tag{36.1}$$

The power-series solution method requires combining the two sums into a single power series in x. We shift the summation index downward by two in the first sum to obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

We can then combine the two sums in (36.1) to obtain

$$\sum_{n=0}^{\infty} \Big((n+2)(n+1) a_{n+2} + a_n \Big) x^n = 0.$$

For the equality to hold, the coefficient of each power of x must vanish separately. We therefore obtain the *recurrence relation*

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \ldots.$$

We observe that even and odd coefficients decouple. We thus obtain two independent sequences starting with first term a_0 or a_1. Developing these sequences, we have for the first sequence,

$$a_0, \quad a_2 = -\frac{1}{2} a_0, \quad a_4 = -\frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0,$$

and so on; and for the second sequence,

$$a_1, \quad a_3 = -\frac{1}{3 \cdot 2} a_1, \quad a_5 = -\frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1,$$

and so on. Using the principle of superposition, the general solution is therefore

$$
\begin{aligned}
y(x) &= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \\
&= a_0 \cos x + a_1 \sin x,
\end{aligned}
$$

as expected.

Problems for Lecture 36

1. Find two independent power series solutions of $y'' - y = 0$. Show that

$$y(x) = a_0 \cosh x + a_1 \sinh x,$$

where

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Solutions to the Problems

Lecture 37 | Series solution of the Airy's equation (Part A)

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We solve by series solution the Airy's equation, an ode that arises in optics, fluid mechanics, and quantum mechanics.

Example: Find the general solution of $y'' - xy = 0$. Notice that there is a non-constant coefficient.

We try the power-series ansatz

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where the a_n coefficients are to be determined. Finding the second derivative by differentiating term-by-term, the ode becomes

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0. \tag{37.1}$$

We shift the summation index of the first sum down by three to obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3}x^{n+1}.$$

When combining the two sums in (37.1), we separate out the extra first term in the first sum. Therefore, (37.1) becomes the single power series

$$2a_2 + \sum_{n=0}^{\infty} \Big((n+3)(n+2)a_{n+3} - a_n \Big) x^{n+1} = 0.$$

Since all the coefficients of powers of x equal zero, we find $a_2 = 0$ and obtain the recursion relation

$$a_{n+3} = \frac{1}{(n+3)(n+2)} a_n.$$

Three sequences of coefficients—those starting with either a_0, a_1 or a_2—decouple. Since $a_2 = 0$, we find immediately that $a_2 = a_5 = a_8 = a_{11} = \cdots = 0$. Starting with a_0, we have

$$a_0, \quad a_3 = \frac{1}{3 \cdot 2} a_0, \quad a_6 = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} a_0,$$

and so on; and starting with a_1,

$$a_1, \quad a_4 = \frac{1}{4 \cdot 3} a_1, \quad a_7 = \frac{1}{7 \cdot 6 \cdot 4 \cdot 3} a_1,$$

and so on. The general solution for $y = y(x)$, can therefore be written as

$$
\begin{aligned}
y(x) &= a_0 \left(1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \dots\right) + a_1 \left(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \dots\right) \\
&= a_0 y_0(x) + a_1 y_1(x).
\end{aligned}
$$

Problems for Lecture 37

1. Find two independent power series solutions of $y'' + xy' - y = 0$. Keep terms up to x^6.

Solutions to the Problems

Lecture 38 | Series solution of the Airy's equation (Part B)

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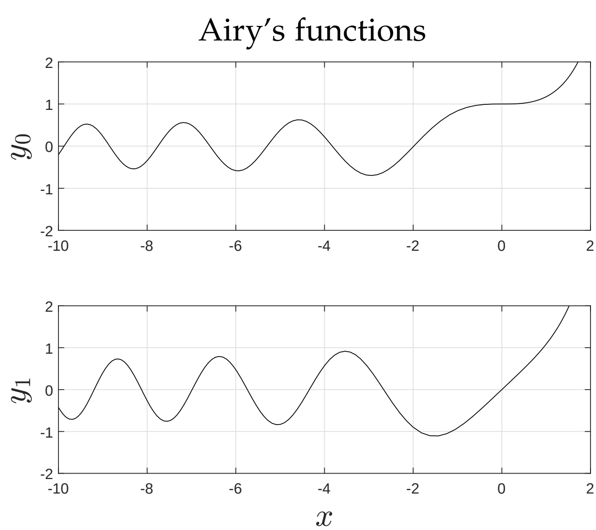
The Airy's equation is given by $y'' - xy = 0$, and the general solution for $y = y(x)$, is given by

$$\begin{aligned} y(x) &= a_0\left(1 + \frac{x^3}{3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 3\cdot 2} + \dots\right) + a_1\left(x + \frac{x^4}{4\cdot 3} + \frac{x^7}{7\cdot 6\cdot 4\cdot 3} + \dots\right) \\ &= a_0 y_0(x) + a_1 y_1(x). \end{aligned}$$

Suppose we would like to graph the solutions $y = y_0(x)$ and $y = y_1(x)$ versus x by solving the differential equation $y'' - xy = 0$ numerically. What initial conditions should we use? Evidently, $y = y_0(x)$ solves the ode with initial values $y(0) = 1$ and $y'(0) = 0$, whereas $y = y_1(x)$ solves the ode with initial values $y(0) = 0$ and $y'(0) = 1$.

The numerical solutions, obtained using MATLAB, are shown in the figure below. Note that the solutions oscillate for negative x and grow exponentially for positive x. This can be understood by recalling that $y'' + y = 0$ has oscillatory sine and cosine solutions and $y'' - y = 0$ has exponential solutions.



Numerical solution of Airy's equation

Problems for Lecture 38

1. Given the two Airy's functions $y = y_0(x)$ and $y = y_1(x)$ as defined by their power series, solve the Airy's equation $y'' - xy = 0$ with the initial conditions $y(0) = 1$ and $y'(0) = 1$.

Solutions to the Problems

Practice Quiz | Series solutions

1. The value of $\cosh^2 t - \sinh^2 t$ is equal to

a) -1

b) 0

c) 1

d) 2

2. The general solution to $y'' + x^2 y = 0$ is given by

a) $y(x) = a_0 \left(1 + \dfrac{x^3}{12} + \dots\right) + a_1 \left(x + \dfrac{x^4}{20} + \dots\right)$

b) $y(x) = a_0 \left(1 - \dfrac{x^3}{12} + \dots\right) + a_1 \left(x - \dfrac{x^4}{20} + \dots\right)$

c) $y(x) = a_0 \left(1 + \dfrac{x^4}{12} + \dots\right) + a_1 \left(x + \dfrac{x^5}{20} + \dots\right)$

d) $y(x) = a_0 \left(1 - \dfrac{x^4}{12} + \dots\right) + a_1 \left(x - \dfrac{x^5}{20} + \dots\right)$

3. The general solution to $y'' - xy' + y = 0$ is given by

a) $y(x) = a_0 \left(1 + \dfrac{x^2}{2} + \dfrac{x^4}{24} + \dots\right) + a_1 x$

b) $y(x) = a_0 \left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + \dots\right) + a_1 x$

c) $y(x) = a_0 \left(1 + \dfrac{x^2}{2} - \dfrac{x^4}{24} + \dots\right) + a_1 x$

d) $y(x) = a_0 \left(1 - \dfrac{x^2}{2} - \dfrac{x^4}{24} + \dots\right) + a_1 x$

Solutions to the Practice quiz

Week V

Systems of Differential Equations

We solve a coupled system of homogeneous first-order differential equations with constant coefficients. This system of odes can be written in matrix form, and we explain how to convert these equations into a standard matrix algebra eigenvalue problem. The two-dimensional solutions are visualized using phase portraits. We then discuss the important application of coupled harmonic oscillators and the calculation of normal modes. The normal modes are those motions for which the individual masses that make up the system oscillate with the same frequency.

Lecture 39 | Systems of homogeneous linear first-order odes

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We consider a system of homogeneous linear odes with constant coefficients given by

$$\dot{x}_1 = ax_1 + bx_2, \quad \dot{x}_2 = cx_1 + dx_2,$$

which can be written in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

or more succinctly as $\dot{\mathrm{x}} = \mathrm{Ax}$.

We take as our ansatz $\mathrm{x}(t) = \mathrm{v}e^{\lambda t}$, where v and λ are independent of t and v is a two-by-one column matrix. Upon substitution into the ode, we obtain

$$\lambda \mathrm{v} e^{\lambda t} = \mathrm{A} \mathrm{v} e^{\lambda t};$$

and upon cancellation of the exponential, we obtain the eigenvalue problem

$$\mathrm{Av} = \lambda \mathrm{v}.$$

The characteristic equation for our two-by-two matrix is given by

$$\det(\mathrm{A} - \lambda \mathrm{I}) = \lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

We will demonstrate how to solve two separate cases: (i) eigenvalues of A are distinct and real; (ii) eigenvalues of A are complex conjugates. The third case for which the eigenvalues are repeated will be omitted. These three cases are analogous to those previously considered when solving the homogeneous constant-coefficient second-order ode.

Problems for Lecture 39

1. Consider the system of homogeneous linear odes with constant coefficients given by

$$\dot{x}_1 = ax_1 + cx_2, \qquad \dot{x}_2 = cx_1 + bx_2.$$

Prove that the eigenvalues of the resulting characteristic equation are real.

Solutions to the Problems

Lecture 40 | Distinct real eigenvalues

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Example: Find the general solution of $\dot{x}_1 = x_1 + x_2, \ \dot{x}_2 = 4x_1 + x_2$.
The equation in matrix form is given by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

or $\dot{\mathrm{x}} = \mathrm{Ax}$. With our ansatz $\mathrm{x}(t) = \mathrm{v}e^{\lambda t}$, we obtain the eigenvalue problem $\mathrm{Av} = \lambda\mathrm{v}$. The characteristic equation of A is given by

$$\det(\mathrm{A} - \lambda\mathrm{I}) = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0,$$

and the eigenvalues are found to be $\lambda_1 = -1$ and $\lambda_2 = 3$. To determine the corresponding eigenvectors, we substitute the eigenvalues successively into $(\mathrm{A} - \lambda\mathrm{I})\mathrm{v} = 0$, and denote the eigenvectors as $\mathrm{v}_1 = \begin{pmatrix} v_{11} & v_{21} \end{pmatrix}^{\mathrm{T}}$ and $\mathrm{v}_2 = \begin{pmatrix} v_{12} & v_{22} \end{pmatrix}^{\mathrm{T}}$.

For $\lambda_1 = -1$, and unknown eigenvector v_1, we obtain from $(\mathrm{A} - \lambda_1\mathrm{I})\mathrm{v}_1 = 0$ the result $v_{21} = -2v_{11}$. Recall that an eigenvector is only unique up to multiplication by a constant: we may therefore take $v_{11} = 1$. For $\lambda_2 = 3$, and unknown eigenvector v_2, we obtain from $(\mathrm{A} - \lambda_2\mathrm{I})\mathrm{v}_2 = 0$ the result $v_{22} = 2v_{12}$. Here, we take $v_{12} = 1$.

Therefore, our eigenvalues and eigenvectors are given by

$$\lambda_1 = -1, \ \mathrm{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \qquad \lambda_2 = 3, \ \mathrm{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Using the principle of superposition, the general solution to the system of odes is given by

$$\mathrm{x}(t) = c_1\mathrm{v}_1 e^{\lambda_1 t} + c_2\mathrm{v}_2 e^{\lambda_2 t};$$

or explicitly writing out the components,

$$x_1(t) = c_1 e^{-t} + c_2 e^{3t}, \qquad x_2(t) = -2c_1 e^{-t} + 2c_2 e^{3t}.$$

Problems for Lecture 40

1. Using matrix algebra, find the general solution of $\dot{x}_1 = -x_2$ and $\dot{x}_2 = -2x_1 - x_2$.

Solutions to the Problems

Lecture 41 | Complex-conjugate eigenvalues

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Example: Find the general solution of $\dot{x}_1 = -\frac{1}{2}x_1 + x_2$ *and* $\dot{x}_2 = -x_1 - \frac{1}{2}x_2$.

The equations in matrix form are

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The ansatz $\mathrm{x} = \mathrm{v}e^{\lambda t}$ leads to the characteristic equation

$$\det\left(\mathrm{A} - \lambda \mathrm{I}\right) = \lambda^2 + \lambda + \frac{5}{4} = 0,$$

which has complex-conjugate roots. We denote the two eigenvalues as

$$\lambda = -\frac{1}{2} + i \quad \text{and} \quad \bar{\lambda} = -\frac{1}{2} - i.$$

The eigenvectors also occur as a complex-conjugate pair, and the eigenvector v associated with the eigenvalue λ satisfies $-iv_1 + v_2 = 0$. Normalizing with $v_1 = 1$, we have $\mathrm{v} = \begin{pmatrix} 1 & i \end{pmatrix}^{\mathrm{T}}$. We have therefore determined two independent complex solutions to the system of odes, that is,

$$\mathrm{v}e^{\lambda t} \quad \text{and} \quad \bar{\mathrm{v}}e^{\bar{\lambda} t},$$

and we can form a linear combination of these two complex solutions to construct two independent real solutions. Namely, the two real solutions are $\mathrm{Re}\{\mathrm{v}e^{\lambda t}\}$ and $\mathrm{Im}\{\mathrm{v}e^{\lambda t}\}$. We have

$$\mathrm{Re}\{\mathrm{v}e^{\lambda t}\} = \mathrm{Re}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix} e^{\left(-\frac{1}{2}+i\right)t}\right\} = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix};$$

and

$$\mathrm{Im}\{\mathrm{v}e^{\lambda t}\} = \mathrm{Im}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix} e^{\left(-\frac{1}{2}+i\right)t}\right\} = e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Taking a linear superposition of these two real solutions yields the general solution to the system of odes, given by

$$\mathrm{x} = e^{-t/2}\left(A\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + B\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right);$$

or explicitly writing out the components,

$$x_1 = e^{-t/2}\left(A\cos t + B\sin t\right), \qquad x_2 = e^{-t/2}\left(-A\sin t + B\cos t\right).$$

Problems for Lecture 41

1. Using matrix algebra, find the general solution of $\dot{x}_1 = x_1 - 2x_2$ and $\dot{x}_2 = x_1 + x_2$.

Solutions to the Problems

Practice Quiz | Systems of differential equations

1. The system of odes given by $\dot{x}_1 = ax_1 + bx_2$ and $\dot{x}_2 = cx_1 + dx_2$ has the same characteristic equation as the second order ode given by

a) $a\ddot{x} + b\dot{x} + cx = 0$

b) $\ddot{x} + (a+b)\dot{x} + (c+d)x = 0$

c) $\ddot{x} - (a+d)\dot{x} + (ad-bc)x = 0$

d) $(a+b)\ddot{x} + (b+c)\dot{x} + (c+d)x = 0$

2. The general solution of $\dot{x}_1 = x_1 + 2x_2$ and $\dot{x}_2 = 2x_1 + x_2$ is given by

a) $x_1 = c_1e^{3t} + c_2e^{-t}$
$x_2 = c_1e^{3t} + c_2e^{-t}$

b) $x_1 = c_1e^{3t} + c_2e^{-t}$
$x_2 = c_1e^{3t} - c_2e^{-t}$

c) $x_1 = c_1e^{3t} + c_2e^{-t}$
$x_2 = c_1e^{-t} + c_2e^{3t}$

d) $x_1 = c_1e^{3t} + c_2e^{-t}$
$x_2 = c_1e^{-t} - c_2e^{3t}$

3. The general solution of $\dot{x}_1 = -2x_1 + x_2$ and $\dot{x}_2 = -x_1 - 2x_2$ is given by

a) $x_1 = e^{-t}(A\cos 2t + B\sin 2t)$
$x_2 = e^{-t}(A\sin 2t + B\cos 2t)$

b) $x_1 = e^{-2t}(A\cos t + B\sin t)$
$x_2 = e^{-2t}(A\sin t + B\cos t)$

c) $x_1 = e^{-t}(A\cos 2t + B\sin 2t)$
$x_2 = e^{-t}(-A\sin 2t + B\cos 2t)$

d) $x_1 = e^{-2t}(A\cos t + B\sin t)$
$x_2 = e^{-2t}(-A\sin t + B\cos t)$

Solutions to the Practice quiz

Lecture 42 | Phase portraits

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The solution of a system of two first-order odes for x_1 and x_2 can be visualized by drawing a phase portrait, with "x-axis" x_1 and "y-axis" x_2. Each curve drawn on the phase portrait corresponds to a different initial condition, and can be viewed as the trajectory of a particle at position (x_1, x_2) moving with a velocity given by $(\dot{x}_1, \dot{x}_2)$.

For a two-by-two system, written as $\dot{\mathrm{x}} = \mathrm{Ax}$, the point $\mathrm{x} = (0, 0)$ is called an equilibrium point, or fixed point, of the system. If x is at the fixed point initially, then x remains there for all time because $\dot{\mathrm{x}} = 0$ at the fixed point. This fixed point, or equilibrium, may be stable or unstable, and the qualitative picture of the phase portrait depends on the stability of the equilibrium, which in turn depends on the eigenvalues of the characteristic equation.

If there are two distinct real eigenvalues of the same sign, we say that the fixed point is a node. When the eigenvalues are both negative the fixed point is a stable node, and when they are both positive, the fixed point is an unstable node. If the eigenvalues have opposite sign, the fixed point is a saddle point.

If there are complex-conjugate eigenvalues, we say that the fixed point is a spiral. A spiral can be stable or unstable and this depends on the sign of the real part of the eigenvalues. If the real part is negative, then the solution decays exponentially and the fixed point corresponds to a stable spiral; if the real part is positive then the solution grows exponentially and the fixed point corresponds to an unstable spiral. Furthermore, a spiral may wind around the fixed point clockwise or counterclockwise, and this so-called handedness of the spiral can be determined by examining the differential equations directly. In the next few lectures, we will present phase portraits representing nodes, saddle points and spirals.

Problems for Lecture 42

1. Determine if the fixed points of the following systems are nodes, saddle points, or spirals, and determine their stability:

a)

$$\dot{x}_1 = -3x_1 + \sqrt{2}x_2, \qquad \dot{x}_2 = \sqrt{2}x_1 - 2x_2;$$

b)

$$\dot{x}_1 = x_1 + x_2, \qquad \dot{x}_2 = 4x_1 + x_2;$$

c)

$$\dot{x}_1 = -\frac{1}{2}x_1 + x_2, \qquad \dot{x}_2 = -x_1 - \frac{1}{2}x_2.$$

Solutions to the Problems

Lecture 43 | Stable and unstable nodes

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When there are two distinct real eigenvalues of the same sign, the fixed point is called a node. Consider the differential equations given by

$$\dot{x}_1 = -3x_1 + \sqrt{2}x_2, \qquad \dot{x}_2 = \sqrt{2}x_1 - 2x_2.$$

An eigenvalue analysis of this system results in eigenvalues and eigenvectors given by

$$\lambda_1 = -4, \quad \mathrm{v}_1 = \begin{pmatrix} 1 \\ -\sqrt{2}/2 \end{pmatrix}; \qquad \lambda_2 = -1, \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix};$$

and the general solution of the system is written as

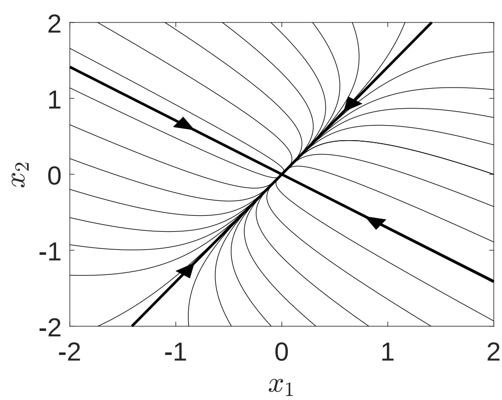
$$\mathrm{x}(t) = c_1\mathrm{v}_1 e^{\lambda_1 t} + c_2\mathrm{v}_2 e^{\lambda_2 t}.$$

Because $\lambda_1, \lambda_2 < 0$, both exponential solutions for $\mathrm{x} = \mathrm{x}(\mathrm{t})$ decay in time and $\mathrm{x} \to (0,0)$ as $t \to \infty$. This is why we say that the node is stable. If both eigenvalues were positive, the node would be unstable.

To draw the phase portrait, one considers the eigenfunctions. On the one hand, suppose initial conditions are such that $c_2 = 0$. The solution $\mathrm{x}(t)$ is then equal to a scalar function of time times the eigenvector v_1. Since x is proportional to v_1, trajectories with $c_2 = 0$ must lie on the line $x_2 = -\sqrt{2}x_1/2$. This line can be drawn on the phase portrait with arrow pointing in towards the origin.

On the other hand, suppose initial conditions are such that $c_1 = 0$. The solution is then equal to a scalar times the eigenvector v_2. Since x is proportional to v_2, trajectories with $c_1 = 0$ must lie on the line $x_2 = \sqrt{2}x_1$. This line can also be drawn on the phase portrait with arrow again pointing in towards the origin.

Because $|\lambda_1| > |\lambda_2|$, the trajectory approaches the fixed point quicker along v_1 than along v_2, resulting in a final approach to the origin along v_2, as is evident in the drawn phase portrait.



Problems for Lecture 43

1. Consider the system of differential equations given by

$$\dot{x}_1 = 2x_1 + x_2, \qquad \dot{x}_2 = x_1 + 2x_2.$$

Determine the eigenvalues and eigenvectors, and sketch the phase portrait.

Solutions to the Problems

Lecture 44 | Saddle points

View this lecture on YouTube

If there are two distinct real eigenvalues of opposite sign, the fixed point is a saddle point. Consider the differential equations given by

$$\dot{x}_1 = x_1 + x_2, \qquad \dot{x}_2 = 4x_1 + x_2.$$

An eigenvalue analysis of this system results in eigenvalues and eigenvectors given by

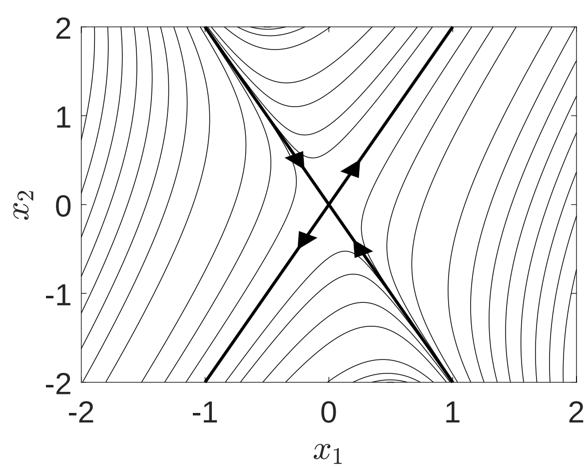
$$\lambda_1 = -1, \quad \mathrm{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \qquad \lambda_2 = 3, \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix};$$

and the general solution is again written as

$$\mathrm{x}(t) = c_1 \mathrm{v}_1 e^{\lambda_1 t} + c_2 \mathrm{v}_2 e^{\lambda_2 t}.$$

Because $\lambda_1 < 0$, trajectories approach the fixed point along the direction of the first eigenvector, and because $\lambda_2 > 0$, trajectories move away from the fixed point along the direction of the second eigenvector. Ultimately, a saddle point is an unstable equilibrium because for any initial conditions such that $c_2 \neq 0$, $|\mathrm{x}(t)| \to \infty$ as $t \to \infty$.

To draw a phase portrait of the solution, we first consider motion along the directions of the eigenvectors. Along the first eigenvector, we have $x_2 = -2x_1$, and this line is to be drawn on the phase portrait with arrow pointing in towards origin. Along the second eigenvector, we have $x_2 = 2x_1$, and this line is to be drawn on the phase portrait with arrow pointing away from the origin. Immediately, we see that the motion is in towards the origin in the direction of the first eigenvector, and away from the origin in the direction of the second eigenvector. The remainder of the phase trajectories can be sketched or computer drawn, and the resulting phase portrait is shown.



Problems for Lecture 44

1. Consider the system of differential equations given by

$$\dot{x}_1 = -x_1 + 3x_2, \qquad \dot{x}_2 = 2x_1 + 4x_2.$$

Determine the eigenvalues and eigenvectors, and sketch the phase portrait.

Solutions to the Problems

Lecture 45 | Spiral points

View this lecture on YouTube

If there are complex conjugate eigenvalues, the fixed point is a spiral point. Consider the system of differential equations given by

$$\dot{x}_1 = -\frac{1}{2}x_1 + x_2, \qquad \dot{x}_2 = -x_1 - \frac{1}{2}x_2.$$

An eigenvalue analysis of this system results in the complex eigenvalue and eigenvector

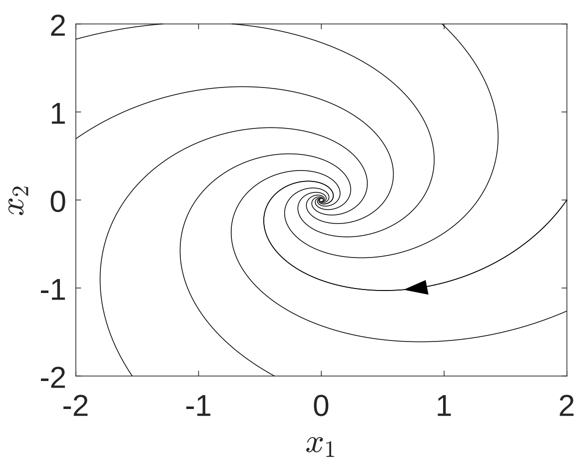
$$\lambda = -\frac{1}{2} + i, \quad \mathrm{v} = \begin{pmatrix} 1 \\ i \end{pmatrix},$$

and their complex conjugates. The general solution is written as

$$\mathrm{x}(t) = e^{-t/2}\left[A\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + B\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right].$$

The trajectories in the phase portrait are spirals centered at the fixed point. If the $\mathrm{Re}\{\lambda\} > 0$ the trajectories spiral out, and if $\mathrm{Re}\{\lambda\} < 0$ they spiral in. The spirals around the fixed point may be clockwise or counterclockwise, depending on the governing equations.

Here, since $\mathrm{Re}\{\lambda\} = -1/2 < 0$, the trajectories spiral into the origin. To determine whether the spiral is clockwise or counterclockwise, we can examine the time derivatives at the point $(x_1, x_2) = (0, 1)$. At this point in the phase space, $(\dot{x}_1, \dot{x}_2) = (1, -1/2)$, and a particle on this trajectory moves to the right and downward, indicating a clockwise spiral. The corresponding phase portrait is shown.



Problems for Lecture 45

1. Consider the system of differential equations given by

$$\dot{x}_1 = x_1 + x_2, \qquad \dot{x}_2 = -x_1 + x_2.$$

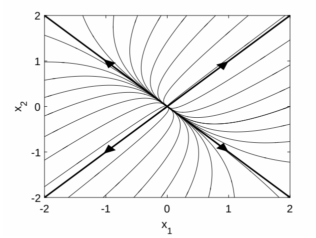
Determine the eigenvalues and eigenvectors, and sketch the phase portrait.

Solutions to the Problems

Practice Quiz | Phase portraits

1. Select the correct phase portrait for the following system of differential equations:

$$\dot{x}_1 = 2x_1 + x_2, \qquad \dot{x}_2 = x_1 + 2x_2.$$



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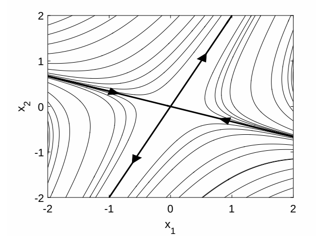
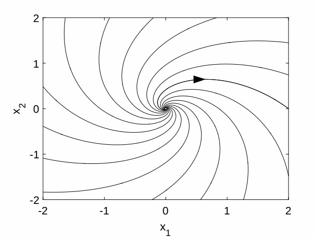


(b)



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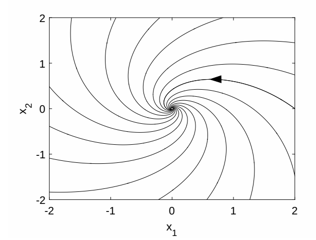
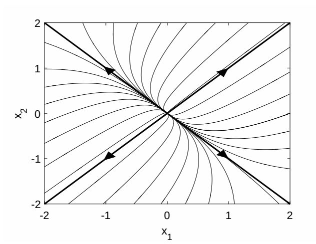


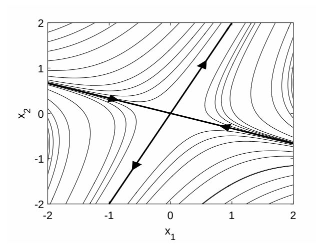
(d)

2. Select the correct phase portrait for the following system of differential equations:

$$\dot{x}_1 = -x_1 + 3x_2, \qquad \dot{x}_2 = 2x_1 + 4x_2.$$



(a)



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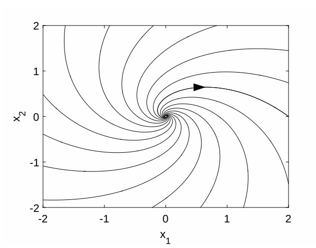

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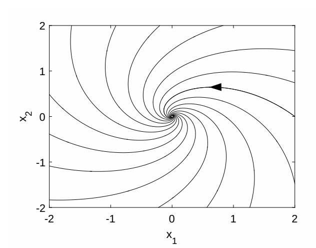
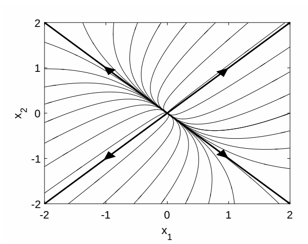


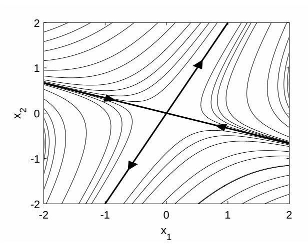
(d)

3. Select the correct phase portrait for the following system of differential equations:

$$\dot{x}_1 = x_1 + x_2, \qquad \dot{x}_2 = -x_1 + x_2.$$



(a)



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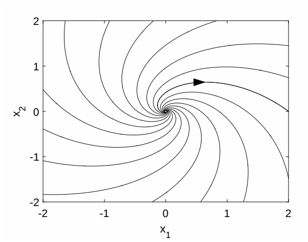

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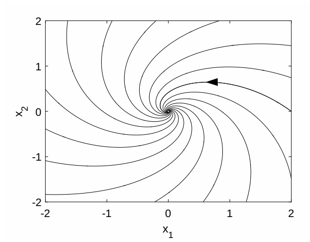

(d)

Solutions to the Practice quiz

Lecture 46 | Coupled oscillators

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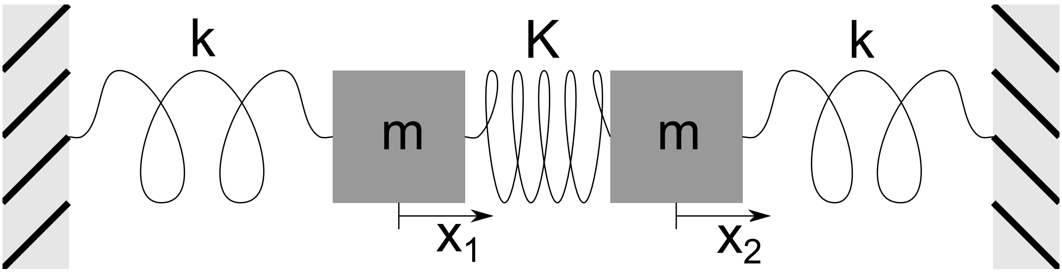


The normal modes of a physical system are oscillations of the entire system that take place at a single frequency. Perhaps the simplest example of a system containing two distinct normal modes is the coupled mass-spring system shown above. We will see that an eigenvector analysis of this system, similar to what we have just done for coupled first-order odes, will exhibit the normal modes and reveal the true nature of the oscillation.

In the figure, the position variables x_1 and x_2 are measured from the equilibrium positions of the masses. Hooke's law states that the spring force is linearly proportional to the extension length of the spring, measured from equilibrium. By considering the extension of each spring and the sign of each force, Newton's law $F = ma$ written separately for each mass is

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - K(x_1 - x_2), \\ m\ddot{x}_2 &= -kx_2 - K(x_2 - x_1). \end{aligned}$$

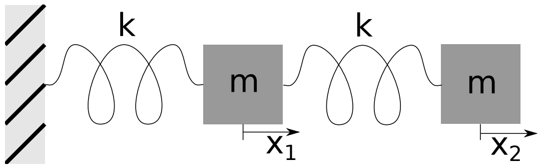
Collecting terms proportional to x_1 and x_2 results in

$$\begin{aligned} m\ddot{x}_1 &= -(k+K)x_1 + Kx_2, \\ m\ddot{x}_2 &= Kx_1 - (k+K)x_2. \end{aligned}$$

The equations for the coupled mass-spring system form a system of two homogeneous linear second-order odes. In matrix form, $m\ddot{\mathrm{x}} = \mathrm{Ax}$, or explicitly,

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -(k+K) & K \\ K & -(k+K) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Problems for Lecture 46



1. Consider the mass-spring system shown above. Determine the matrix equation represented as $m\ddot{\mathrm{x}} = \mathrm{A}\mathrm{x}$.

Solutions to the Problems

Lecture 47 | Normal modes (eigenvalues)

View this lecture on YouTube

We now solve the two masses, three springs system with governing equation

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -(k+K) & K \\ K & -(k+K) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We try the ansatz $\mathrm{x} = \mathrm{v}e^{rt}$, and obtain the eigenvalue problem $\mathrm{Av} = \lambda\mathrm{v}$, with $\lambda = mr^2$. The eigenvalues are determined by solving the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda + k + K)^2 - K^2 = 0;$$

and the two solutions for λ are given by

$$\lambda_1 = -k, \quad \lambda_2 = -(k+2K).$$

The corresponding values of r in our ansatz $\mathrm{x} = \mathrm{v}e^{rt}$, with $r = \pm\sqrt{\lambda/m}$, are

$$r_1 = i\sqrt{k/m}, \quad \bar{r}_1, \quad r_2 = i\sqrt{(k+2K)/m}, \quad \bar{r}_2.$$

Since the values of r are pure imaginary, we know that $x_1(t)$ and $x_2(t)$ will oscillate with angular frequencies $\omega_1 = \mathrm{Im}\{r_1\}$ and $\omega_2 = \mathrm{Im}\{r_2\}$, that is,

$$\omega_1 = \sqrt{k/m}, \quad \omega_2 = \sqrt{(k+2K)/m}.$$

These are the so-called frequencies of the two normal modes. A normal mode with angular frequency ω_i is periodic with period $T_i = 2\pi/\omega_i$. In the next lecture, we find the eigenvectors and physically intepret the results.

Lecture 48 | Normal modes (eigenvectors)

View this lecture on YouTube

We are solving the two masses, three springs system with governing equation

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -(k+K) & K \\ K & -(k+K) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We tried the ansatz $\mathrm{x} = \mathrm{v}e^{rt}$, and obtained the eigenvalue problem $\mathrm{Av} = \lambda\mathrm{v}$, with $\lambda = mr^2$. The eigenvalues were found to be

$$\lambda_1 = -k, \quad \lambda_2 = -(k+2K).$$

The eigenvectors, or so-called normal modes of oscillations, correspond to the mass-spring system oscillating at a single frequency. The eigenvector with eigenvalue λ_1 satisfies

$$-Kv_{11} + Kv_{12} = 0,$$

so that $v_{11} = v_{12}$. The normal mode with frequency $\omega_1 = \sqrt{k/m}$ thus follows a motion where $x_1 = x_2$. During this motion the center spring length does not change, and the frequency is independent of K.

Next, we determine the eigenvector with eigenvalue λ_2:

$$Kv_{21} + Kv_{22} = 0,$$

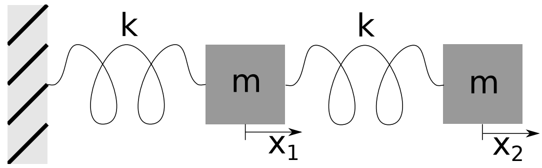
so that $v_{21} = -v_{22}$. The normal mode with frequency $\omega_2 = \sqrt{(k+2K)/m}$ thus follows a motion where $x_1 = -x_2$. During this motion the two equal masses symmetrically push or pull against each side of the middle spring.

A general solution for $\mathrm{x}(t)$ can be constructed from the eigenvalues and eigenvectors. Our ansatz was $\mathrm{x} = \mathrm{v}e^{rt}$, and for each of two eigenvectors v, we have a pair of complex conjugate values for r. Accordingly, we first apply the principle of superposition to obtain four real solutions, and then apply the principle again to obtain the general solution. With $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{(k+2K)/m}$, the general solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(A\cos\omega_1 t + B\sin\omega_1 t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}(C\cos\omega_2 t + D\sin\omega_2 t),$$

where the now real constants A, B, C, and D can be determined from the four independent initial conditions, $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$, and $\dot{x}_2(0)$.

Problems for Lecture 48



1. Consider the matrix equation obtained from the mass-spring system shown above. Find the angular frequencies and the eigenvectors of the two normal modes.

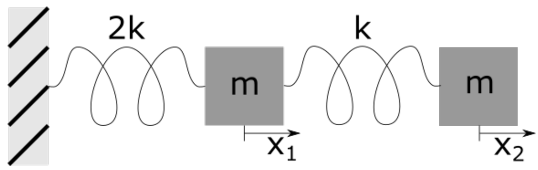
Solutions to the Problems

Practice Quiz | Normal modes

1. The matrix equation for the pictured mass-spring system is given by

a) $m\dfrac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -k & k \\ k & -k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

b) $m\dfrac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



c) $m\dfrac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3k & k \\ k & -k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

d) $m\dfrac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4k & k \\ k & -k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

2. The angular frequencies of the normal modes for the mass-spring system of Question 1 are approximately given by

a) $\omega_1 = 0.77\sqrt{\dfrac{k}{m}}, \quad \omega_2 = 1.85\sqrt{\dfrac{k}{m}}$

b) $\omega_1 = 0.84\sqrt{\dfrac{k}{m}}, \quad \omega_2 = 2.14\sqrt{\dfrac{k}{m}}$

c) $\omega_1 = 1.45\sqrt{\dfrac{k}{m}}, \quad \omega_2 = 2.34\sqrt{\dfrac{k}{m}}$

d) $\omega_1 = 1.82\sqrt{\dfrac{k}{m}}, \quad \omega_2 = 2.66\sqrt{\dfrac{k}{m}}$

3. The eigenvectors of the normal modes for the mass-spring system of Question 1 are approximately given by

a) $\mathrm{v}_1 = \begin{pmatrix} 1 \\ 1.32 \end{pmatrix}, \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ -0.22 \end{pmatrix}$

b) $\mathrm{v}_1 = \begin{pmatrix} 1 \\ 1.98 \end{pmatrix}, \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ -0.35 \end{pmatrix}$

c) $\mathrm{v}_1 = \begin{pmatrix} 1 \\ 2.41 \end{pmatrix}, \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ -0.41 \end{pmatrix}$

d) $\mathrm{v}_1 = \begin{pmatrix} 1 \\ 2.92 \end{pmatrix}, \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ -0.63 \end{pmatrix}$

Solutions to the Practice quiz

Week VI

Partial Differential Equations

To solve a partial differential equation, we must first define the Fourier series, and the Fourier sine and cosine series. We then derive the one-dimensional diffusion equation, which is a partial differential equation for the time-evolution of the concentration of a dye over one spatial dimension. We proceed to solve this equation using the method of separation of variables. This method yields a mathematical solution for a dye diffusing length-wise within a finite length pipe.

Lecture 49 | Fourier series

View this lecture on YouTube

Fourier series will be needed to solve the diffusion equation. A periodic function $f(x)$ with period $2L$ can be represented as a Fourier series in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{49.1}$$

Determination of the coefficients $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$ makes use of orthogonality relations for sine and cosine. We first define the Kronecker delta δ_{nm} as

$$\delta_{nm} = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{otherwise.} \end{cases}$$

The orthogonality relations for n and m positive integers are then given as the integration formulas

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L\delta_{nm}, \quad \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L\delta_{nm},$$
$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0.$$

To determine the coefficient a_n, we multiply both sides of (49.1) by $\cos(n\pi x/L)$, change the dummy summation variable from n to m, and integrate over x from $-L$ to L. To determine the coefficient b_n, we do the same except multiply by $\sin(n\pi x/L)$. To illustrate the procedure, when finding b_n, integration will result in

$$\begin{aligned} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx &= \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx \\ &= \sum_{m=1}^{\infty} b_m \left(L\delta_{nm}\right) \\ &= L b_n, \end{aligned}$$

which can be solved for b_n. The final results for the coefficients are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

Problems for Lecture 49

1. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

a) Show that $f(x + 2L) = f(x)$, that is, $f(x)$ is a periodic function with period $2L$.

b) Show that a_0 is twice the average value of $f(x)$.

Solutions to the Problems

Lecture 50 | Fourier sine and cosine series

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We now know that the Fourier series of a periodic function with period $2L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

The Fourier series simplifies if $f(x)$ is an even function such that $f(-x) = f(x)$, or an odd function such that $f(-x) = -f(x)$. We make use of the following facts. The function $\cos(n\pi x/L)$ is an even function and $\sin(n\pi x/L)$ is an odd function. The product of two even functions is an even function. The product of two odd functions is an even function. The product of an even and an odd function is an odd function. And if $g(x)$ is an even function, then

$$\int_{-L}^{L} g(x)\, dx = 2 \int_{0}^{L} g(x)\, dx;$$

and if $g(x)$ is an odd function, then

$$\int_{-L}^{L} g(x)\, dx = 0.$$

Regarding the Fourier series, then, if $f(x)$ is even we have

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \qquad b_n = 0;$$

and the Fourier series for an even function is given by the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad f(x) \text{ even}.$$

If $f(x)$ is odd, then

$$a_n = 0, \qquad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\, dx;$$

and the Fourier series for an odd function is given by the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad f(x) \text{ odd}.$$

Problems for Lecture 50

1. Show that the Fourier series for an odd function satisfies $f(0) = 0$ and that the Fourier series for an even function satisfies $f'(0) = 0$.

Solutions to the Problems

Lecture 51 | Fourier series (example)

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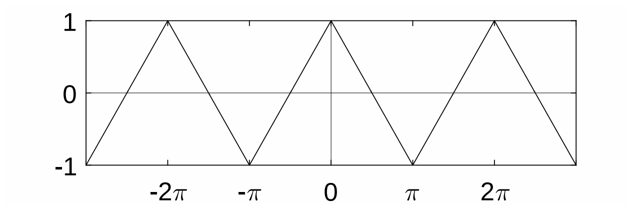
Example: Determine the Fourier series of the triangle wave, shown in the following figure:



The triangle wave

Evidently, the triangle wave is an even function of x with period 2π, and its definition over half a period is

$$f(x) = 1 - \frac{2x}{\pi}, \quad 0 < x < \pi.$$

Because $f(x)$ is even, it can be represented by a Fourier cosine series (with $L = \pi$) given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ with } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx.$$

The coefficient $a_0/2$ is the average value of $f(x)$ over one period, which is clearly zero. The coefficients a_n for $n > 0$ are

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\, dx = \frac{4}{n^2\pi^2}(1 - \cos n\pi) = \begin{cases} 8/(n^2\pi^2), & \text{if } n \text{ odd;} \\ 0, & \text{if } n \text{ even.} \end{cases}$$

The Fourier cosine series for the triangle wave is therefore given by

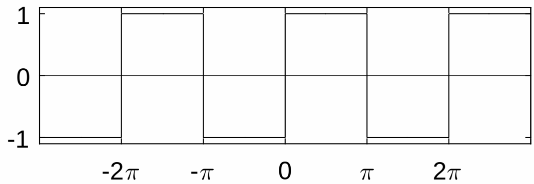
$$f(x) = \frac{8}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right).$$

Convergence of this series is rapid. As an interesting aside, evaluation of this series at $x = 0$, using $f(0) = 1$, yields in an infinite series for π^2, which is usually written in the form

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots.$$

Problems for Lecture 51

1. Determine the Fourier series of the square wave, shown in the following figure:



The square wave

Solutions to the Problems

Practice Quiz | Fourier series

1. The Fourier series of a periodic function of period $2L$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$. The value of $f(0)$ is given by

a) $\frac{a_0}{2}$

b) $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$

c) $\sum_{n=1}^{\infty} b_n$

d) $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n + b_n)$

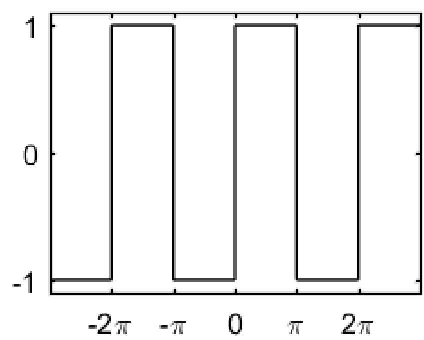
2. The Fourier series of the square wave in the shown figure is given by

a) $f(x) = \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \dots\right)$

b) $f(x) = \frac{4}{\pi}\left(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots\right)$

c) $f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$

d) $f(x) = \frac{4}{\pi}\left(\sin x - \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \dots\right)$



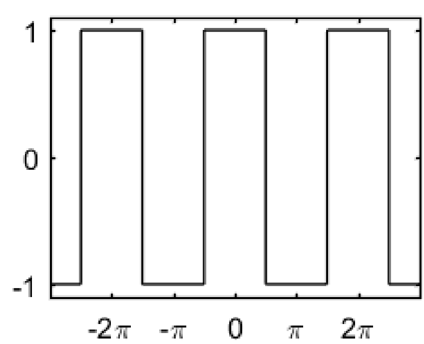
3. The Fourier series of the square wave in the shown figure is given by

a) $g(x) = \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \dots\right)$

b) $g(x) = \frac{4}{\pi}\left(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots\right)$

c) $g(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$

d) $g(x) = \frac{4}{\pi}\left(\sin x - \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \dots\right)$



Solutions to the Practice quiz

Lecture 52 | The diffusion equation

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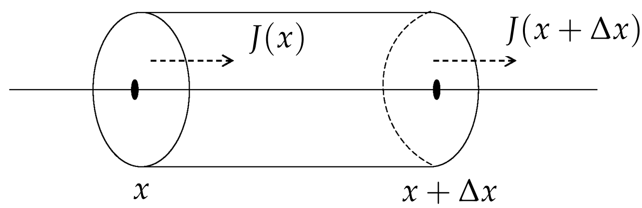
To derive the diffusion equation in one spacial dimension, we imagine a still liquid in a pipe of constant cross sectional area. A small quantity of dye is placed uniformly across a cross section of the pipe and allowed to diffuse up and down the pipe. We define $u(x,t)$ to be the concentration (mass per unit length) of the dye at position x along the pipe, and our goal is to find the equation satisfied by u.



The mass of dye at time t in a pipe volume located between x and $x+\Delta x$ is given to order Δx by

$$M = u(x,t)\Delta x.$$

This mass of dye diffuses in and out of the cross sectional ends as shown above. We assume the rate of diffusion is proportional to the concentration gradient, a relationship known as Fick's first law of diffusion. Fick's law assumes the mass flux J (mass per unit time) across a cross section of the pipe is given by

$$J = -Du_x,$$

where $D > 0$ is the diffusion constant, and we have used the notation $u_x = \partial u/\partial x$. The mass flux is opposite in sign to the gradient of concentration so that the flux is from high concentration to low concentration. The time rate of change in the mass of dye is given by the difference between the mass flux in and the mass flux out of the cross sectional volume. A positive mass flux signifies diffusion from left to right. Therefore, the time rate of change of the dye mass is given by

$$\frac{dM}{dt} = J(x,t) - J(x+\Delta x,t);$$

or rewriting in terms of $u(x,t)$,

$$u_t(x,t)\Delta x = -Du_x(x,t) + Du_x(x+\Delta x,t) = D\left(u_x(x+\Delta x,t) - u_x(x,t)\right).$$

Dividing by Δx and taking the limit $\Delta x \to 0$ results in the diffusion equation, given by

$$u_t = Du_{xx}.$$

Problems for Lecture 52

1. Consider the one-dimensional diffusion equation in a pipe of length L. Nondimensionalize the diffusion equation using L as the unit of length and L^2/D as the unit of time.

Solutions to the Problems

Lecture 53 | Solution of the diffusion equation (separation of variables)

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We solve the diffusion equation for a dye in a pipe of length L with diffusion constant D,

$$u_t = Du_{xx},$$

where $u(x,t)$ is the dye concentration. We will use the method of *separation of variables*. We assume that $u(x,t)$ can be written as a product of two other functions, one dependent only on position x and the other dependent only on time t. That is, we make the ansatz,

$$u(x,t) = X(x)T(t).$$

We will find all possible solutions of this type—there will be an infinite number—and apply the principle of superposition to combine them to construct a general solution.

Substitution of our ansatz into the diffusion equation results in

$$XT' = DX''T,$$

which we rewrite by separating the x and t dependence to opposite sides of the equation:

$$\frac{X''}{X} = \frac{1}{D}\frac{T'}{T}.$$

The left-hand side of this equation is independent of t and the right-hand side is independent of x. Both sides of this equation are therefore independent of both t and x and must be equal to a constant, which we call $-\lambda$. Introduction of this *separation constant* results in the two ordinary differential equations

$$X'' + \lambda X = 0, \quad T' + \lambda DT = 0. \tag{53.1}$$

To proceed further, boundary conditions need to be specified at the pipe ends. We will assume here that the ends open up into large reservoirs of clear fluid so that $u(0,t) = u(L,t) = 0$ for all times. Applying these boundary conditions to our ansatz results in

$$u(0,t) = X(0)T(t) = 0, \quad u(L,t) = X(L)T(t) = 0.$$

Since these boundary conditions are valid for all t, we must have $X(0) = X(L) = 0$. These are called homogeneous Dirichlet boundary conditions. In the next lecture, we proceed to solve for $X = X(x)$.

Problems for Lecture 53

1. If the ends of the pipe were closed preventing a further diffusion of dye, then the mass flux through the pipe's ends would be zero, and the appropriate boundary conditions would be $u_x(0,t) = u_x(L,t) = 0$ for all times. Determine the appropriate boundary conditions on $X = X(x)$.

Solutions to the Problems

Lecture 54 | Solution of the diffusion equation (eigenvalues)

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We now solve the equation for $X = X(x)$ with homogeneous Dirichlet boundary conditions:

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0.$$

Clearly, the trivial solution $X(x) = 0$ is a solution, and we will see that nontrivial solutions exist only for discrete values of λ. These discrete values and the corresponding functions $X = X(x)$ are called the eigenvalues and eigenfunctions of the ode.

The form of the general solution of the ode depends on the sign of λ. One can show that nontrivial solutions exist only when $\lambda > 0$. We therefore write $\lambda = \mu^2$, and determine the general solution of

$$X'' + \mu^2 X = 0$$

to be

$$X(x) = A\cos\mu x + B\sin\mu x.$$

The boundary condition at $x = 0$ results in $A = 0$, and the boundary condition at $x = L$ results in

$$B\sin\mu L = 0.$$

The solution $B = 0$ results in the unsought trivial solution. Therefore, we must have

$$\sin\mu L = 0,$$

which is an equation for μ, with solutions given by $\mu_n = n\pi/L$, where n is a nonzero integer. We have thus determined all the unique eigenvalues $\lambda = \mu^2 > 0$ to be

$$\lambda_n = (n\pi/L)^2, \quad n = 1, 2, 3, \ldots,$$

with corresponding eigenfunctions

$$X_n = \sin(n\pi x/L).$$

There is no need here to include the multiplication by an arbitrary constant, which will be added later.

Problems for Lecture 54

1. Show that the equations

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0$$

have no nontrivial solutions for $\lambda \leq 0$.

2. Solve the following equation for $X = X(x)$ with given boundary conditions:

$$X'' + \lambda X = 0, \quad X'(0) = X'(L) = 0.$$

Solutions to the Problems

Practice Quiz | Separable partial differential equations

1. The units of the diffusion constant D in the diffusion equation $u_t = Du_{xx}$ are

a) lt^2

b) lt^{-2}

c) l^2t

d) l^2t^{-1}

2. The one-dimensional wave equation is given by $u_{tt} = c^2u_{xx}$. With $u(x,t) = X(x)T(t)$, the separated ordinary differential equations can be written as

a) $X' + \lambda X = 0,\ T' + \lambda c^2 T = 0$

b) $X'' + \lambda X = 0,\ T' + \lambda c^2 T = 0$

c) $X' + \lambda X = 0,\ T'' + \lambda c^2 T = 0$

d) $X'' + \lambda X = 0,\ T'' + \lambda c^2 T = 0$

3. The eigenvalues and eigenfunctions of the differential equation $X'' + \lambda X = 0$, with mixed boundary conditions $X(0) = 0$ and $X'(L) = 0$, are given by

a) $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n = \sin\left(\frac{n\pi x}{L}\right),\ n = 1, 2, 3, \ldots$

b) $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n = \cos\left(\frac{n\pi x}{L}\right),\ n = 0, 1, 2, 3, \ldots$

c) $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad X_n = \sin\left(\frac{(2n-1)\pi x}{2L}\right),\ n = 1, 2, 3, \ldots$

d) $\lambda_0 = 0, \quad X_0 = 1,$
$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad X_n = \cos\left(\frac{(2n-1)\pi x}{2L}\right),\ n = 1, 2, 3, \ldots$

Solutions to the Practice quiz

Lecture 55 | Solution of the diffusion equation (Fourier series)

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With the eigenvalues $\lambda_n = (n\pi/L)^2$, the differential equation for $T = T(t)$ becomes

$$T' + \left(n^2\pi^2 D/L^2\right) T = 0,$$

which has solution proportional to

$$T_n = e^{-n^2\pi^2 Dt/L^2}.$$

Therefore, with ansatz $u(x,t) = X(x)T(t)$ and eigenvalues λ_n, we conclude that the functions

$$u_n(x,t) = \sin(n\pi x/L)e^{-n^2\pi^2 Dt/L^2}$$

satisfy the diffusion equation and the spatial boundary conditions for every positive integer n.

The principle of linear superposition for homogeneous linear differential equations then states that the general solution to the diffusion equation with the spatial boundary conditions is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)e^{-n^2\pi^2 Dt/L^2}.$$

The final solution step is to determine the unknown coefficients b_n by satisfying conditions on the initial dye concentration. We assume that $u(x,0) = f(x)$, where $f(x)$ is some specific function defined on $0 \leq x \leq L$. At $t = 0$, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L).$$

We immediately recognize this equation as a Fourier sine series for an odd function $f(x)$ with period $2L$. A Fourier sine series results because of the boundary condition $f(0) = 0$. From our previous solution for the coefficients of a Fourier sine series, we determine that

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Problems for Lecture 55

1. Determine the solution to the diffusion equation $u_t = Du_{xx}$ for the concentration $u = u(x,t)$ with closed pipe ends corresponding to the boundary conditions $u_x(0) = u_x(L) = 0$, and with general initial conditions given by $u(x,0) = f(x)$. Use the results of the previously solved problems.

Solutions to the Problems

Lecture 56 | Diffusion equation (example)

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Example: Determine the concentration of a dye in a pipe of length L, where the dye is initially concentrated uniformly across the center of the pipe with total mass M_0, and the ends of the pipe open onto a large reservoir.

We solve the diffusion equation with homogeneous Dirichlet boundary conditions, and model the initial concentration of the dye by a delta-function centered at $x = L/2$, that is, $u(x, 0) = M_0\delta(x - L/2)$. The Fourier sine series coefficients are therefore given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L M_0\delta(x - \frac{L}{2})\sin\frac{n\pi x}{L}\,dx \\ &= \frac{2M_0}{L}\sin(n\pi/2) \\ &= \begin{cases} 2M_0/L & \text{if } n = 1, 5, 9, \dots; \\ -2M_0/L & \text{if } n = 3, 7, 11, \dots; \\ 0 & \text{if } n = 2, 4, 6, \dots. \end{cases} \end{aligned}$$

With b_n determined, the solution for $u(x, t)$ is given by

$$u(x, t) = \frac{2M_0}{L}\sum_{n=0}^{\infty}(-1)^n \sin\left(\frac{(2n+1)\pi x}{L}\right)e^{-(2n+1)^2\pi^2 Dt/L^2}.$$

When $t \gg L^2/D$, the leading-order term in the series is a good approximation and is given by

$$u(x, t) \approx \frac{2M_0}{L}\sin(\pi x/L)e^{-\pi^2 Dt/L^2}.$$

The mass of the dye in the pipe is decreasing in time, diffusing into the reservoirs located at both ends. The total mass in the pipe at time t can be found from

$$M(t) = \int_0^L u(x, t)dx,$$

and when $t \gg L^2/D$, we have

$$M(t) = \frac{4M_0}{\pi}e^{-\pi^2 Dt/L^2}.$$

Problems for Lecture 56

1. Determine the concentration of a dye in a pipe of length L at position x and time t, where the dye is initially concentrated uniformly across the center of the pipe with total mass M_0, and the ends of the pipe are closed.

Solutions to the Problems

Practice Quiz | The diffusion equation

1. The solution of $T' + \lambda D T = 0$ with eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ results in the eigenfunctions

a) $T_n = \cos\left(\frac{n\pi D t}{L}\right)$

b) $T_n = \sin\left(\frac{n\pi D t}{L}\right)$

c) $T_n = \exp\left(\frac{n^2\pi^2 D t}{L^2}\right)$

d) $T_n = \exp\left(-\frac{n^2\pi^2 D t}{L^2}\right)$

2. If $u(x,t) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)\exp(-n^2\pi^2 D t/L^2)$ and $u(x,0) = f(x)$, the general formula for a_n is given by

a) $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$

b) $a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$

c) $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{\pi^2 D}{L^2}\right) dx$

d) $a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{\pi^2 D}{L^2}\right) dx$

3. Suppose that a pipe with open ends has an initial dye concentration centered at one-quarter of the pipe's length. The long-time $t >> L^2/D$ solution for the concentration is given by

a) $u(x,t) = \frac{\sqrt{2}M_0}{2L}\sin\left(\frac{\pi x}{L}\right)\exp\left(-\frac{\pi^2 D t}{L^2}\right)$

b) $u(x,t) = \frac{\sqrt{2}M_0}{L}\sin\left(\frac{\pi x}{L}\right)\exp\left(-\frac{\pi^2 D t}{L^2}\right)$

c) $u(x,t) = \frac{2M_0}{L}\sin\left(\frac{\pi x}{L}\right)\exp\left(-\frac{\pi^2 D t}{L^2}\right)$

d) $u(x,t) = \frac{2\sqrt{2}M_0}{L}\sin\left(\frac{\pi x}{L}\right)\exp\left(-\frac{\pi^2 D t}{L^2}\right)$

Solutions to the Practice quiz

Appendices

Appendix A | Complex numbers

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We define the imaginary unit i to be one of the two numbers (the other being $-i$) that satisfies $z^2 + 1 = 0$. Formally, we write $i = \sqrt{-1}$. A complex number z and its complex conjugate $\bar{z}$ are written as

$$z = x + iy, \qquad \bar{z} = x - iy,$$

where x and y are real numbers. We call x the real part of z and y the imaginary part, and write

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z.$$

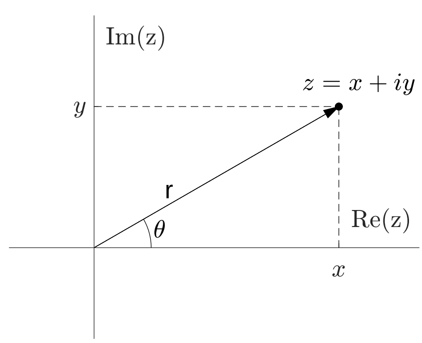
A linear combination of z and $\bar{z}$ can be used to construct the real and imaginary parts of a number:

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im} z = \frac{1}{2i}(z - \bar{z}).$$

We can add, subtract and multiply complex numbers (using $i^2 = -1$) to get new complex numbers. Division of two complex numbers is simplified by writing $z/w = z\overline{w}/w\overline{w}$. The exponential function of a complex number can be determined from a Taylor series. We have

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) = \cos\theta + i\sin\theta. \end{aligned}$$

The complex number $x + iy = z$ can be represented in the complex plane with $\operatorname{Re} z$ as the x-axis and $\operatorname{Im} z$ as the y-axis (see the figure). Then with $x = r\cos\theta$ and $y = r\sin\theta$, we have $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$.



Appendix B | Nondimensionalization

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To nondimensionalize an equation, first determine its base dimensions. These may be one or several of the fundamental dimensions such as time, length, mass and charge. The Buckingham π theorem states that after nondimensionalization, the remaining number of dimensionless parameters is the number of dimensional parameters minus the number of base dimensions. In practice, nondimensionalizing an equation can transform equations arising from seemingly different physical problems into an identical dimensionless equation, and can also significantly reduce the number of parameters that must be explored in a numerical solution.

To illustrate nondimensionalization, we consider the pendulum equation given by

$$ml\ddot{\theta} + cl\dot{\theta} + mg\sin\theta = F_0\cos\omega t,$$

where the four terms of the equation are called the inertial term, the frictional force, the restoring force, and the external force, respectively. The base dimensions are time (t), length (l) and mass (m). The angle θ expressed in radians is dimensionless, since it is defined as the ratio of two lengths (arc length divided by radius). The dimensional parameters are m, l, c, g, F_0, and ω, for a total of six. The Buckingham π theorem says that after nondimensionalizing the equation, there should be no more than $6-3=3$ dimensionless parameters.

Each term in this equation must have the same dimension, and from the first term we see that each term must have dimension ml/t^2, which is the units of force. The frictional parameter denoted by c must have units m/t. The gravitational acceleration g has units of l/t^2.

If we divide the pendulum equation by ml, and define the natural frequency of the oscillator without damping to be $\omega_0 = \sqrt{g/l}$, then the pendulum equation becomes

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \omega_0^2\sin\theta = \frac{F_0}{ml}\cos\omega t;$$

and the units of the various grouping of parameters are given by

$$[\frac{c}{m}] = 1/t, \quad [\omega_0] = 1/t, \quad [\frac{F_0}{ml}] = 1/t^2, \quad [\omega] = 1/t.$$

At this stage there are four dimensional parameters and one base dimension, and nondimensionalization requires only the choice of a time scale. We see that there is some freedom in the choice of time scale, and we first proceed without losing generality by defining the dimensionless time by $\tau = t/t_*$, where the time scale t_* will be chosen later from our dimensional parameters.

Using $d/dt = (d\tau/dt)d/d\tau = t_*^{-1}d/d\tau$ and $d^2/dt^2 = t_*^{-2}d^2/d\tau^2$, and multiplying the equation by t_*^2, the pendulum equation transforms to

$$\frac{d^2\theta}{d\tau^2} + \frac{ct_*}{m}\frac{d\theta}{d\tau} + \omega_0^2 t_*^2 \sin\theta = \frac{F_0 t_*^2}{ml}\cos(\omega t_* \tau).$$

A standard choice for t_* arises from equating the coefficient of the inertial term to the coefficient of the restoring force, that is, $1 = \omega_0^2 t_*^2$. This defines $t_* = 1/\omega_0$. The resulting dimensionless equation becomes

$$\frac{d^2\theta}{d\tau^2} + \alpha\frac{d\theta}{d\tau} + \sin\theta = \gamma\cos\beta\tau,$$

where the three dimensionless parameters are given by

$$\alpha = \frac{c}{m\omega_0}, \quad \beta = \frac{\omega}{\omega_0}, \quad \gamma = \frac{F_0}{mg}.$$

Another choice for nondimensionalization is to equate the coefficient of the frictional term and the coefficient of the restoring force, that is, $ct_*/m = \omega_0^2 t_*^2$. This defines $t_* = c/(m\omega_0^2)$. The resulting dimensionless equation after multiplication by $m^2\omega_0^2/c^2$ becomes

$$\alpha\frac{d^2\theta}{d\tau^2} + \frac{d\theta}{d\tau} + \sin\theta = \gamma\cos\beta\tau,$$

where here

$$\alpha = \frac{m^2 g}{lc^2}, \quad \beta = \frac{lc\omega}{mg}, \quad \gamma = \frac{F_0}{mg}.$$

This latter dimensionless equation is useful when studying very large damping effects, corresponding to the limit $\alpha \to 0$. The resulting governing equation then becomes first-order in time.

Appendix C | Matrices and determinants

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A two-by-two matrix A, with two rows and two columns, can be written as

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The first row has elements a_{11} and a_{12}; the second row has elements a_{21} and a_{22}. The first column has elements a_{11} and a_{21}; the second column has elements a_{12} and a_{22}. Matrices can be multiplied by scalars and added. This is done element-by-element and the straightforward definitions are

$$k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}, \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}.$$

Matrices can also be multiplied. Matrix multiplication does not commute, and two matrices can be multiplied only if the number of columns of the matrix on the left equals the number of rows of the matrix on the right. One multiplies matrices by going across the rows of the first matrix and down the columns of the second matrix. The two-by-two example is given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}.$$

A system of linear algebraic equations can be written in matrix form. For instance, with a_{ij} and b_i given numbers, and x_i unknowns, the (two-by-two) system of equations given by

$$a_{11}x_1 + a_{12}x_2 = b_1, \qquad a_{21}x_1 + a_{22}x_2 = b_2,$$

can be written in matrix form as $\mathrm{Ax} = \mathrm{b}$, or explicitly as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

A unique solution to $\mathrm{Ax} = \mathrm{b}$ exists only when $\det \mathrm{A} \neq 0$, and it is possible to write this unique solution as $\mathrm{x} = \mathrm{A}^{-1}\mathrm{b}$, where A^{-1} is the inverse matrix satisfying $\mathrm{A}^{-1}\mathrm{A} = \mathrm{A}\mathrm{A}^{-1} = \mathrm{I}$. Here, I is the identity matrix satisfying $\mathrm{AI} = \mathrm{IA} = \mathrm{A}$. For the two-by-two matrix, the determinant is given by the product of the diagonal elements minus the product of the off-diagonal elements:

$$\det \mathrm{A} = a_{11}a_{22} - a_{12}a_{21}.$$

Appendix D | Eigenvalues and eigenvectors

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The eigenvalue problem for an n-by-n matrix A is given by

$$\mathrm{Ax} = \lambda \mathrm{x},$$

where the scalar λ is called the eigenvalue and the n-by-1 column vector x is called the eigenvector. Using the identity matrix I, we can rewrite the eigenvalue equation as

$$(\mathrm{A} - \lambda \mathrm{I})\mathrm{x} = 0.$$

When A is a two-by-two matrix, then

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{A} - \lambda \mathrm{I} = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix}.$$

A solution other than $\mathrm{x} = 0$ of the eigenvalue equation exists provided

$$\det(\mathrm{A} - \lambda \mathrm{I}) = 0.$$

This equation is called the characteristic equation of A, and is given by

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0,$$

which is usually written as

$$\lambda^2 - \mathrm{Tr}\mathrm{A}\,\lambda + \det\mathrm{A} = 0,$$

where TrA is the trace of the matrix A, equal to the sum of the diagonal elements. The eigenvalues can be real and distinct, complex conjugates, or repeated.

After determining an eigenvalue, say $\lambda = \lambda_1$, the corresponding eigenvector v_1 can be found by solving

$$(\mathrm{A} - \lambda_1 \mathrm{I})\mathrm{v}_1 = 0,$$

or explicitly in the two-by-two case,

$$\begin{pmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The second equation represented above is always a mutiple of the first equation, and the eigenvector is unique only up to multiplication by a constant.

Appendix E | Partial derivatives

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For a function $f = f(x, y)$ of two variables, the partial derivative of f with respect to x is defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

and similarly for the partial derivative of f with respect to y. To take the partial derivative of f with respect to x, say, take the derivative of f with respect to x holding y fixed. As an example, consider

$$f(x, y) = 2x^3y^2 + y^3.$$

We have

$$\frac{\partial f}{\partial x} = 6x^2y^2, \quad \frac{\partial f}{\partial y} = 4x^3y + 3y^2.$$

Second derivatives are defined as the derivatives of the first derivatives, so we have

$$\frac{\partial^2 f}{\partial x^2} = 12xy^2, \quad \frac{\partial^2 f}{\partial y^2} = 4x^3 + 6y;$$

and the mixed second partial derivatives are independent of the order in which the derivatives are taken:

$$\frac{\partial^2 f}{\partial x \partial y} = 12x^2y, \quad \frac{\partial^2 f}{\partial y \partial x} = 12x^2y.$$

Partial derivatives are necessary for applying the chain rule. Consider $df = f(x + dx, y + dy) - f(x, y)$. We can write df as

$$df = [f(x + dx, y + dy) - f(x, y + dy)] + [f(x, y + dy) - f(x, y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

If one has $f = f(x(t), y(t))$, say, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

And if one has $f = f(x(r, \theta), y(r, \theta))$, say, then

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}.$$

Appendix F | Table of Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. $e^{at} f(t)$	$F(s-a)$
2. 1	$\frac{1}{s}$
3. e^{at}	$\frac{1}{s-a}$
4. t^n	$\frac{n!}{s^{n+1}}$
5. $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
6a. $\sin bt$	$\frac{b}{s^2+b^2}$
6b. $\sinh bt$	$\frac{b}{s^2-b^2}$
7a. $\cos bt$	$\frac{s}{s^2+b^2}$
7b. $\cosh bt$	$\frac{s}{s^2-b^2}$
8. $e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
9. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
10. $t \sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
11. $t \cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
12. $u_c(t)$	$\frac{e^{-cs}}{s}$
13. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
14. $\delta(t-c)$	e^{-cs}
15. $\dot{x}(t)$	$sX(s) - x(0)$
16. $\ddot{x}(t)$	$s^2 X(s) - sx(0) - \dot{x}(0)$

Table of Laplace Transforms

Appendix G | Problem and practice quiz solutions

Solutions to the Practice quiz: Classify differential equations

1. c, d, g. Third order, ordinary, nonlinear.

2. b, c , f. Second order, ordinary, nonlinear.

3. b, d , f. Second order, partial, nonlinear.

4. b, c , e. Second order, ordinary, linear.

5. b, d , e. Second order, partial, linear.

Solutions to the Problems for Lecture 2

1. The modified Euler method is

$$k_1 = \Delta x f(x_n, y_n), \quad k_2 = \Delta x f(x_n + \Delta x, y_n + k_1), \quad y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2).$$

The midpoint method is

$$k_1 = \Delta x f(x_n, y_n), \quad k_2 = \Delta x f(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_1), \quad y_{n+1} = y_n + k_2.$$

Solutions to the Problems for Lecture 3

1.

a) $\dfrac{1}{y}\,dy = \dfrac{x^2-4}{x+4}\,dx$

b) $e^y \cos y\,dy = e^x(1+x)\,dx$

c) $\dfrac{y+1}{y}\,dy = \dfrac{x}{(x+1)}\,dx$

d) $\dfrac{1}{\sin\theta}\,d\theta = -dt$

Solutions to the Problems for Lecture 4

1. Separate the variables, integrate, and solve for the dependent variable.

a) $\displaystyle\int_1^y \frac{dy}{y^{1/2}} = 4\int_0^x x\,dx; \quad 2y^{1/2}\Big|_1^y = 2x^2\Big|_0^x; \quad y^{1/2} - 1 = x^2; \quad y = (1+x^2)^2.$

b) $\displaystyle\int_{x_0}^x \frac{dx}{x(1-x)} = \int_0^t dt.$ Use $\displaystyle\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}.$ $\displaystyle[\ln x - \ln(1-x)]\Big|_{x_0}^x = t;$
$\displaystyle\ln\frac{x(1-x_0)}{x_0(1-x)} = t; \quad \frac{x(1-x_0)}{x_0(1-x)} = e^t; \quad x = \frac{x_0}{x_0 + (1-x_0)e^{-t}}.$

Solutions to the Practice quiz: Separable first-order odes

1. c. $\displaystyle\frac{dy}{dx} = x^{1/2}y^{1/2}, \quad y(1) = 0.$

$$\int_0^y \frac{dy}{y^{1/2}} = \int_1^x x^{1/2}\,dx; \quad 2y^{1/2} = \frac{2}{3}(x^{3/2}-1); \quad y = \frac{(x^{3/2}-1)^2}{9}.$$

2. a. $\displaystyle x\frac{dy}{dx} = y^2, \quad y(1) = 1.$

$$\int_1^y \frac{dy}{y^2} = \int_1^x \frac{dx}{x}; \quad -\left(\frac{1}{y} - 1\right) = \ln x; \quad \frac{1}{y} = 1 - \ln x; \quad y = \frac{1}{1-\ln x}.$$

3. b. $\displaystyle\frac{dy}{dx} = -(\sin x)y, \quad y(\pi/2) = 1.$

$$\int_1^y \frac{dy}{y} = -\int_{\pi/2}^x \sin x\,dx; \quad \ln y = \cos x; \quad y = e^{\cos x}.$$

Solutions to the Problems for Lecture 5

1.

a) $\displaystyle\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin x}{x};$

b) $\displaystyle\frac{dy}{dx} + y = x.$

2. Let $z = 1/x$. Then $x = 1/z$ and $dx/dt = (dx/dz)(dz/dt) = -(1/z^2)dz/dt$. The differential equation becomes $-(1/z^2)dz/dt = (1/z)(1-(1/z))$, and after multiplying by $-z^2$, $dz/dt = -z + 1$. In standard form, the linear ode is $dz/dt + z = 1$.

Solutions to the Problems for Lecture 6

1.

a) $\dfrac{dy}{dx} = x - y, \quad y(0) = -1.$

$dy/dx + y = x; \quad \mu = e^{\int_0^x dx} = e^x;$

$$y = e^{-x}(-1 + \textstyle\int_0^x xe^x\,dx) = e^{-x}(-1 + xe^x\big|_0^x - \int_0^x e^x\,dx) = e^{-x}(-1 + xe^x - e^x + 1)$$
$$= x - 1.$$

b) $\dfrac{dy}{dx} = 2x(1-y), \quad y(0) = 0.$

$dy/dx + 2xy = 2x; \quad \mu = e^{\int_0^x 2x\,dx} = e^{x^2};$

$$y = e^{-x^2}\textstyle\int_0^x 2xe^{x^2}dx = e^{-x^2}\int_0^{x^2} e^u\,du = e^{-x^2}(e^{x^2} - 1) = 1 - e^{-x^2}.$$

Solutions to the Practice quiz: Linear first-order odes

1. d. $(1+x^2)y' + 2xy = 2x, \quad y(0) = 0,$

$$\frac{d}{dx}\left[(1+x^2)y\right] = 2x,\ (1+x^2)y\big|_0^x = \int_0^x 2x\,dx = x^2,\ (1+x^2)y = x^2,\ y = \frac{x^2}{1+x^2}.$$

2. b. $x^2y' + 2xy = 1, \quad y(1) = 2,$

$$\frac{d}{dx}\left[x^2y\right] = 1,\ x^2y\big|_1^x = x^2y - 2 = \int_1^x dx = x - 1,\ x^2y = 1 + x,\ y = \frac{1+x}{x^2}.$$

3. d. $y' + \lambda y = a, \quad y(0) = 0,$

$$\mu = e^{\lambda x},\ y = e^{-\lambda x}\int_0^x ae^{\lambda x}dx = \frac{ae^{-\lambda x}}{\lambda}\left(e^{\lambda x} - 1\right) = \tfrac{a}{\lambda}\left(1 - e^{-\lambda x}\right).$$

Solutions to the Problems for Lecture 7

1. Use $S(t) = S_0e^{rt} + \dfrac{k}{r}e^{rt}(1 - e^{-rt})$, with the unit of time in years, and money in US$. Here, $t = 40$, $S_0 = 0$, $r = 0.06$, and $S(t) =$1,000,000. Solving for k using $S_0 = 0$, we have

$$k = \frac{rS(t)}{e^{rt} - 1} = \frac{0.06 \times 1{,}000{,}000}{e^{0.06\times 40} - 1} = \$5{,}986 \text{ per year}.$$

The total amount saved is (40 years) × ($5,986 per year) ≈ $240,000, and the amount passively earned from the investment is $760,000, about three times as much that was saved.

2. Use $S(t) = S_0 e^{rt} + \frac{k}{r} e^{rt}(1 - e^{-rt})$, with the unit of time in years, and money in US\$. Here, $S(t)$ is the amount owed the bank at time t, S_0 is the total amount borrowed, $r = 0.04$ is the annual interest rate, and $k = -1500 \times 12$ is the annual amount of repayment. We need to solve for S_0 under the condition that $S(T) = 0$ when $T = 30$ (loan repaid after 30 years). We have $0 = S_0 e^{rT} + \frac{k}{r} e^{rT}(1 - e^{-rT})$, or $S_0 = -\frac{k}{r}(1 - e^{-rT}) = \frac{1500 \times 12}{0.04}(1 - e^{-0.04 \times 30}) = \$314,463 \approx \$315,000$.

Solutions to the Problems for Lecture 8

1. Use

$$v(t) = -\frac{mg}{k}(1 - e^{-kt/m}),$$

where $mg/k = 200\,\mathrm{km/hr}$ and $m = 100\,\mathrm{kg}$. With

$$g = (9.8\,\mathrm{m/s^2})(10^{-3}\,\mathrm{km/m})(60\,\mathrm{s/min})^2(60\,\mathrm{min/hr})^2 = 127,008\,\mathrm{km/hr^2},$$

we find $k = 63,504\,\mathrm{kg/hr}$. One-half of the terminal speed for free-fall ($100\,\mathrm{km/hr}$) is therefore attained when

$$v(t) = -\frac{mg}{2k},$$

or

$$(1 - e^{-kt/m}) = 1/2,$$

or $t = m \ln 2/k \approx 4\,\mathrm{sec}$. For 95% of the terminal speed ($190\,\mathrm{km/hr}$), we have

$$(1 - e^{-kt/m}) = 0.95,$$

and this equation is satisfied when $t \approx 17\,\mathrm{sec}$.

Solutions to the Problems for Lecture 9

1. The current $i = dq/dt$, with $q = CV_C$. During charging, $V_C(t) = \mathcal{E}\left(1 - e^{-t/RC}\right)$ and during discharging $V_C(t) = \mathcal{E} e^{-t/RC}$. With $i = C\,dV_C/dt$, we have during charging, $i = (\mathcal{E}/R)e^{-t/RC}$, and during discharging, $i = -(\mathcal{E}/R)e^{-t/RC}$. The currents strengths are the same during both processes but are in opposite directions.

Solutions to the Practice quiz: Applications

1. a. The relevant formula is $S(T) = \frac{k}{r} e^{rT}(1 - e^{-rT})$, where $S(T)$ is the amount in the account after T years, r is the annual return, and k is the annual deposit rate. Solving for k, we have $k = \frac{rS(T)e^{-rT}}{1 - e^{-rT}}$. Substituting $r = 0.1$, $S(T) = 1{,}000{,}000$, $T = 40$, we find $k = 1{,}865.74$. The total saved is $40k = 74{,}629.44 \approx \$75{,}000$.

2. c. Use $v(t) = -\frac{mg}{k}(1 - e^{-kt/m})$, where $mg/k = 200\,\text{km/hr}$, $m = 100\,\text{kg}$ and $v(t) = -150\,\text{km/hr}$. With $g = 127{,}008\ \text{km/hr}^2$, we find $k = 63{,}504\ \text{kg/hr}$. Three-quarter of the terminal speed for free-fall (150 km/hr) is therefore attained when $(1 - e^{-kt/m}) = 3/4$, or $t = m\ln 4/k = 7.86\,\text{sec} \approx 8\,\text{sec}$.

3. c. Use $V_C = \mathcal{E}\left(1 - e^{-t/RC}\right)$. Solve for t to find $t = -RC\ln\left(1 - \frac{V_C}{\mathcal{E}}\right)$. With $R = 3000$ Ω, $C = 0.001$ F, $V_C/\mathcal{E} = 0.95$, we find $t = 8.99\,\text{s} \approx 9\,\text{s}$.

Solutions to the Problems for Lecture 10

1. $\dot{\theta} = u, \quad \dot{u} = -u/q - \sin\theta + f\cos\omega t$

2.

$$\begin{aligned} k_1 &= \Delta t f(t_n, x_n, y_n), & l_1 &= \Delta t g(t_n, x_n, y_n) \\ k_2 &= \Delta t f(t_n + \Delta t, x_n + k_1, y_n + l_1), & l_2 &= \Delta t g(t_n + \Delta t, x_n + k_1, y_n + l_1) \end{aligned}$$

$$x_{n+1} = x_n + \frac{1}{2}(k_1 + k_2), \quad y_{n+1} = y_n + \frac{1}{2}(l_1 + l_2).$$

Solutions to the Problems for Lecture 11

1. With $x = x_h(t) + x_p(t)$, we compute

$$\begin{aligned} \ddot{x} + p(t)\dot{x} + q(t)x &= (\ddot{x}_h + \ddot{x}_p) + p\left(\dot{x}_h + \dot{x}_p\right) + q\left(x_h + x_p\right) \\ &= (\ddot{x}_h + p\dot{x}_h + qx_h) + (\ddot{x}_p + p\dot{x}_p + qx_p) \\ &= 0 + g(t) \\ &= g(t), \end{aligned}$$

since x_h and x_p were assumed to be solutions of the homogeneous and inhomogeneous equations, respectively. Therefore, their sum is a solution of the inhomogeneous equation.

Solutions to the Problems for Lecture 12

1. With $X_1(t) = \exp(\alpha t)$ and $X_2(t) = \exp(\beta t)$, we have

$$W = (\exp(\alpha t))(\beta\exp(\beta t)) - (\alpha\exp(\alpha t))(\exp(\beta t)) = (\beta - \alpha)\exp((\alpha + \beta)t),$$

which is nonzero provided $\alpha \neq \beta$.

Solutions to the Problems for Lecture 13

1.

a) $r^2 - 1 = 0$, so that $r = \pm 1$;

b) $r^2 + 1 = 0$, so that $r = \pm i$;

c) $r^2 - 2r + 1 = 0$, so that $r = 1$.

Solutions to the Practice quiz: Theory of ode

1. c. Functions that differ by a multiplying constant have a zero Wronskian. In particular, $e^{t-t_0} = (e^{-t_0})e^t$ and $\sin(t - \pi) = -\sin t$. However, $\sin(t - \pi/2) = -\cos t$ which has a nonzero Wronskian with $\sin t$.

2. d. The principle of superposition applies so that we can multiply solutions by constants and add them to obtain a solution. The only proposed function that does not follow the principle of superposition is the product of two solutions.

3. c. An ode given by $a\ddot{x} + b\dot{x} + cx = 0$ has complex-conjugate roots if $b^2 - 4ac < 0$. In all four equations, $b^2 - 4ac = 1 - 4c$ and $c = \pm 1$, so only the equations with $c = 1$ have complex-conjugate roots.

Solutions to the Problems for Lecture 14

1. The characteristic equation is $r^2 + 4r + 3 = (r+1)(r+3) = 0$, with roots $r_1 = -1$ and $r_2 = -3$. The general solution is $x(t) = c_1 e^{-t} + c_2 e^{-3t}$. The derivative is $\dot{x}(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$. Initial conditions are satisfied by solving $c_1 + c_2 = 1$ and $-c_1 - 3c_2 = 0$. Solution is $c_1 = 3/2$ and $c_2 = -1/2$. The final solution is $x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} = \frac{3}{2}e^{-t}(1 - \frac{1}{3}e^{-2t})$.

2. The characteristic equation is $r^2 - 1 = 0$, with roots $r_1 = 1$ and $r_2 = -1$. The general solution is $x(t) = c_1 e^t + c_2 e^{-t}$. The derivative is $\dot{x}(t) = c_1 e^t - c_2 e^{-t}$. Initial conditions are satisfied by solving $c_1 + c_2 = x_0$ and $c_1 - c_2 = u_0$. Solution is $c_1 = (x_0 + u_0)/2$ and $c_2 = (x_0 - u_0)/2$. The final solution is

$$x(t) = \left(\frac{x_0 + u_0}{2}\right)e^t + \left(\frac{x_0 - u_0}{2}\right)e^{-t} = x_0\left(\frac{e^t + e^{-t}}{2}\right) + u_0\left(\frac{e^t - e^{-t}}{2}\right).$$

We can define

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2},$$

and write the solution as

$$x(t) = x_0 \cosh t + u_0 \sinh t.$$

The function $\cosh t$ is called the hyperbolic cosine of t, and is pronounced 'cosh tee', where the 'o' sounds like the o in open. The function $\sinh t$ is called the hyperbolic sine of t and is pronounced 'sinsh tee', where the i sounds like the i in inch.

Solutions to the Problems for Lecture 16

1. The characteristic equation is $r^2 - 2r + 5 = 0$, with roots $r_1 = 1 + 2i$ and $r_2 = \bar{r}_1$. The general solution is

$$x(t) = e^t \left(A \cos 2t + B \sin 2t\right), \quad \dot{x}(t) = e^t \left(A \cos 2t + B \sin 2t\right) + 2e^t \left(-A \sin 2t + B \cos 2t\right).$$

The initial conditions $x(0) = 1$ yields $A = 1$ and $\dot{x}(0) = 0$ yields $A + 2B = 0$, or $B = -1/2$. The solution is

$$x(t) = e^t \left(\cos 2t - \frac{1}{2} \sin 2t\right).$$

2. The characteristic equation is $r^2 + 1 = 0$, with roots $r_1 = i$ and $r_2 = -i$. The general solution is $x(t) = A \cos t + B \sin t$. The derivative is $\dot{x}(t) = -A \sin t + B \cos t$. Initial conditions are satisfied by $A = x_0$ and $B = u_0$. The final solution is $x(t) = x_0 \cos t + u_0 \sin t$.

Solutions to the Problems for Lecture 18

1. The characteristic equation is $r^2 - 2r + 1 = (r-1)^2 = 0$, with repeated root $r = 1$. The general solution is $x(t) = (c_1 + c_2 t)e^t$. The derivative is $\dot{x}(t) = (c_1 + c_2(1+t))e^t$. Initial conditions are satisfied by solving $c_1 = 1$ and $c_1 + c_2 = 0$. Solution is $c_1 = 1$ and $c_2 = -1$. Final solution is $x(t) = (1-t)e^t$.

Solutions to the Practice quiz: Homogeneous equations

1. b. The characteristic equation is $r^2 - 3r + 2 = 0$, with roots $r_1 = 2, r_2 = 1$. The general solution is $x(t) = c_1 e^{2t} + c_2 e^t$, with derivative $\dot{x}(t) = 2c_1 e^{2t} + c_2 e^t$. Initial conditions are satisfied by solving $x(0) = c_1 + c_2 = 1$ and $\dot{x}(0) = 2c_1 + c_2 = 0$ to find $c_1 = -1, c_2 = 2$. The final solution is $x(t) = -e^{2t} + 2e^t = -e^{2t}(1 - 2e^{-t})$.

2. b. The characteristic equation is $r^2 - 2r + 2 = 0$, with complex roots $r_\pm = 1 \pm i$. The general solution is $x(t) = e^t \left(A \cos t + B \sin t\right)$, with derivative $\dot{x}(t) = e^t \left((A+B) \cos t + (B-A) \sin t\right)$. Initial conditions are satisfied by $A = 1$ and $A + B = 0$, or $B = -1$. The final solution is $x(t) = e^t \left(\cos t - \sin t\right)$.

3. a. The characteristic equation is $r^2 + 2r + 1 = 0$, with repeated root $r = -1$. The general solution is $x(t) = e^{-t}(c_1 + c_2 t)$, with derivative $\dot{x}(t) = e^{-t}(c_2 - c_1 - c_2 t)$. Initial conditions are satisfied by $c_1 = 0$ and $c_2 - c_1 = 1$, or $c_2 = 1$. The final solution is $x(t) = te^{-t}$.

Solutions to the Problems for Lecture 19

1. Substituting $x(t) = x_h(t) + x_{p_1}(t) + x_{p_2}(t)$ into the inhomogeneous ode, we obtain

$$\begin{aligned}
\ddot{x} + p\dot{x} + qx &= \frac{d^2}{dt^2}(x_h + x_{p_1} + x_{p_2}) + p\frac{d}{dt}(x_h + x_{p_1} + x_{p_2}) + q(x_h + x_{p_1} + x_{p_2}) \\
&= (\ddot{x}_h + p\dot{x}_h + qx_h) + (\ddot{x}_{p_1} + p\dot{x}_{p_1} + qx_{p_1}) + (\ddot{x}_{p_2} + p\dot{x}_{p_2} + qx_{p_2}) \\
&= 0 + g_1 + g_2 \\
&= g_1 + g_2,
\end{aligned}$$

so that the sum of the homogeneous and the two particular solutions solve the ode, and the two free constants in x_h can be used to satisfy the two initial conditions.

Solutions to the Problems for Lecture 20

1. The homogeneous equation has characteristic equation $r^2 + 5r + 6 = (r+2)(r+3) = 0$, with roots $r_1 = -2$ and $r_2 = -3$. Therefore, $x_h = c_1 e^{-2t} + c_2 e^{-3t}$. To find the particular solution, we try the ansatz $x = Ae^{-t}$. The resulting equation is $A - 5A + 6A = 1$, with solution $A = 1/2$. The general solution is therefore

$$x = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{2}e^{-t}, \qquad \dot{x} = -2c_1 e^{-2t} - 3c_2 e^{-3t} - \frac{1}{2}e^{-t}.$$

Satisfying the initital conditions results in the two equations $c_1 + c_2 = -1/2$ and $-2c_1 - 3c_2 = 1/2$, with solution $c_1 = -1$ and $c_2 = 1/2$. The final solution is

$$x(t) = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t} = \frac{1}{2}e^{-t}\left(1 - 2e^{-t} + e^{-2t}\right).$$

Solutions to the Practice quiz: Solving inhomogeneous equations

1. a. For all three questions, we need to solve $\ddot{x} + 5\dot{x} + 6x = 2e^{-t}$. The homogeneous equation has characteristic equation given by $r^2 + 5r + 6 = (r+2)(r+3) = 0$, with roots $r_1 = -2$ and $r_2 = -3$. The general homogeneous solution is therefore $x_h = c_1 e^{-2t} + c_2 e^{-3t}$. To find a particular solution, we try $x_p = Ae^{-t}$. Substitution into the inhomogeneous ode yields $A - 5A + 6A = 2$, or $A = 1$. Therefore, the general solution to the inhomogeneous ode is the sum of the general homogeneous solution and the particular solution, or

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t} + e^{-t}.$$

The three questions then entail solving for c_1 and c_2.

Here $x(0) = 0$ and $\dot{x}(0) = 0$. We obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= -1 \\ -2c_1 - 3c_2 &= 1, \end{aligned}$$

with solution $c_1 = -2$ and $c_2 = 1$. Therefore,

$$x(t) = -2e^{-2t} + e^{-3t} + e^{-t} = e^{-t}(1 - 2e^{-t} + e^{-2t}).$$

2. a. Here $x(0) = 1$ and $\dot{x}(0) = 0$. We obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ -2c_1 - 3c_2 &= 1, \end{aligned}$$

with solution $c_1 = 1$ and $c_2 = -1$. Therefore,

$$x(t) = e^{-2t} - e^{-3t} + e^{-t} = e^{-t}(1 + e^{-t} - e^{-2t}).$$

3. a. Here $x(0) = 0$ and $\dot{x}(0) = 1$. We obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= -1 \\ -2c_1 - 3c_2 &= 2, \end{aligned}$$

with solution $c_1 = -1$ and $c_2 = 0$. Therefore,

$$x(t) = -e^{-2t} + e^{-t} = e^{-t}(1 - e^{-t}).$$

Solutions to the Problems for Lecture 22

1.

a) Try $x(t) = A\cos t + B\sin t$. Equating the coefficients of cosine and sine, we obtain $5A + 3B = -2$ and $3A - 5B = 0$. The solution is $A = -5/17$ and $B = -3/17$.

b) Solve $\ddot{z} - 3\dot{z} - 4z = 2e^{it}$. Try $z(t) = Ce^{it}$. Determine $C = (-5 + 3i)/17$ and find

$$x_p = Re\{z_p\} = \text{Re}\left\{\frac{1}{17}(-5 + 3i)(\cos t + i\sin t)\right\} = -\frac{1}{17}(5\cos t + 3\sin t).$$

Solutions to the Problems for Lecture 23

1. Try $x = At + B$. Equating the coefficients of t^1 and t^0, we obtain $A = 1$ and $A + B = 0$, or $B = -1$. The solution is $x_p = t - 1$.

Solutions to the Practice quiz: Particular solutions

1. b. Try $x_p = Ae^{2t}$. Equating the coefficients of the exponential function, we obtain $4A + 6A + 2A = 2$, with solution $A = 1/6$. The particular solution is given by $x_p(t) = \frac{1}{6}e^{2t}$.

2. d. Solve the complex differential equation $\ddot{z} - \dot{z} - 2z = 2e^{2it}$, and take $x_p = \text{Re}\{z_p\}$. Try $z_p = Ce^{2it}$. Equating the coefficients of the exponential function, we obtain $-4C - 2iC - 2C = 2$, or $C = \dfrac{-3+i}{10}$. We find $x_p(t) = \dfrac{1}{10}\text{Re}\{(-3+i)(\cos 2t + i \sin 2t)\}$ $= -\dfrac{1}{10}(3\cos 2t + \sin 2t)$.

3. c. Try $x_p = At + B$ to obtain $-3A + 2At + 2B = t + 1$. Equating coefficients of powers of t, we find $2A = 1$ and $-3A + 2B = 1$, or $A = \frac{1}{2}, B = \frac{5}{4}$. The particular solution is given by $x_p(t) = \frac{1}{2}t + \frac{5}{4}$.

Solutions to the Problems for Lecture 24

1. The homogeneous equation has characteristic equation $r^2 + 3r + 2 = (r+1)(r+2) = 0$, with roots $r_1 = -1$ and $r_2 = -2$. Therefore, $x_h = c_1e^{-t} + c_2e^{-2t}$. To find the particular solution, we try the ansatz $x = Ate^{-2t}$. The resulting equation yields $A = -1$. The general solution is therefore

$$x = c_1e^{-t} + (c_2 - t)e^{-2t}, \qquad \dot{x} = -c_1e^{-t} + (-1 - 2c_2 + 2t)e^{-2t}.$$

Satisfying the initial conditions results in the two equations $c_1 + c_2 = 0$ and $-c_1 - 2c_2 - 1 = 0$, with solution $c_1 = 1$ and $c_2 = -1$. The final solution is

$$x(t) = e^{-t} - (1+t)e^{-2t}.$$

Solutions to the Problems for Lecture 25

1. Write the RLC circuit equation as

$$\frac{L}{R}\frac{d^2q}{dt^2} + \frac{dq}{dt} + \frac{1}{RC}q = \frac{\mathcal{E}_0}{R}\cos\omega t.$$

To remove the coefficient of the restoring force, we define the dimensionless time τ to be

$$\tau = t/RC.$$

After multiplication by RC, the RLC circuit equation becomes

$$\frac{L}{R^2C}\frac{d^2q}{d\tau^2} + \frac{dq}{d\tau} + q = \mathcal{E}_0 C \cos \omega RC\tau.$$

To remove the coefficient of the inhomogeneous term, which has units of charge, we can define the dimensionless charge Q to be

$$Q = \frac{q}{\mathcal{E}_0 C},$$

and the resulting dimensionless equation becomes

$$\alpha\frac{d^2Q}{d\tau^2} + \frac{dQ}{d\tau} + Q = \cos \beta\tau,$$

with

$$\alpha = \frac{L}{R^2C}, \quad \beta = \omega RC.$$

Solutions to the Problems for Lecture 26

1. Write the mass on a spring equation as

$$\frac{m}{c}\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{k}{c}x = \frac{F_0}{c}\cos \omega t.$$

To remove the coefficient of the restoring force in the final dimensionless equation, we define the dimensionless time τ to be

$$\tau = kt/c.$$

After multiplication by c/k, the mass on a spring equation becomes

$$\frac{mk}{c^2}\frac{d^2x}{d\tau^2} + \frac{dx}{d\tau} + x = \frac{F_0}{k}\cos \frac{\omega c}{k}\tau.$$

To remove the coefficient of the inhomogeneous term, we can define the dimensionless position X to be

$$X = \frac{kx}{F_0},$$

and the dimensionless equation becomes

$$\alpha\frac{d^2X}{d\tau^2} + \frac{dX}{d\tau} + X = \cos \beta\tau,$$

with
$$\alpha = \frac{mk}{c^2}, \quad \beta = \frac{\omega c}{k}.$$

Solutions to the Problems for Lecture 28

1. The long-time solution corresponds to the particular solution. We consider the complex ode
$$\ddot{z} + \alpha\dot{z} + z = e^{i\beta t},$$
with $x_p = \mathrm{Re}\{z_p\}$. With the ansatz $z_p = Ae^{i\beta t}$, we have $-\beta^2 A + i\alpha\beta A + A = 1$, or
$$A = \frac{1}{(1-\beta^2) + i\alpha\beta} = \frac{(1-\beta^2) - i\alpha\beta}{(1-\beta^2)^2 + \alpha^2\beta^2}.$$
To determine the amplitude of the resulting oscillation, we make use of the polar form of a complex number and write
$$(1-\beta^2) - i\alpha\beta = \sqrt{(1-\beta^2)^2 + \alpha^2\beta^2}\, e^{i\phi},$$
where $\tan\phi = -\alpha\beta/(1-\beta^2)$. The long-time solution is given by
$$\begin{aligned} x_p &= \mathrm{Re}\left\{ \frac{\sqrt{(1-\beta^2)^2 + \alpha^2\beta^2}}{(1-\beta^2)^2 + \alpha^2\beta^2} e^{i(\beta t + \phi)} \right\} \\ &= \frac{1}{\sqrt{(1-\beta^2)^2 + \alpha^2\beta^2}} \cos(\beta t + \phi); \end{aligned}$$
and the amplitude of the resulting oscillation is therefore given by
$$\text{amplitude} = \frac{1}{\sqrt{(1-\beta^2)^2 + \alpha^2\beta^2}}.$$

Solutions to the Practice quiz: Applications and resonance

1. d. We solve $\ddot{x} + \dot{x} = 1$, with $x(0) = 0$ and $\dot{x}(0) = 0$. The homogeneous equation has characteristic equation given by $r^2 + r = r(r+1) = 0$, with roots $r_1 = 0$ and $r_2 = -1$. The general homogeneous solution is therefore $x_h = c_1 + c_2 e^{-t}$. A particular solution is easily guessed to be $x_p = t$. The general solution to the inhomogeneous ode is therefore
$$x(t) = c_1 + c_2 e^{-t} + t.$$
With $x(0) = 0$ and $\dot{x}(0) = 0$, we obtain the system of equations
$$\begin{aligned} c_1 + c_2 &= 0 \\ c_2 &= 1, \end{aligned}$$

so that $c_1 = -1$ and $c_2 = 1$. Therefore,

$$x(t) = -1 + e^{-t} + t = (t-1) + e^{-t}.$$

2. b. We solve $\ddot{x} - x = \cosh t$, with $x(0) = 0$ and $\dot{x}(0) = 0$. The hyperbolic sine and cosine functions satisfy $\frac{d}{dt}\cosh t = \sinh t$ and $\frac{d}{dt}\sinh t = \cosh t$. Furthermore, $\sinh(0) = 0$ and $\cosh(0) = 1$. The general homogeneous solution is $x_h = A\cosh t + B\sinh t$. Since the inhomogeneous term is a solution of the homogeneous equation, we try as our particular solution $x_p = Ct\cosh t + Dt\sinh t$. The derivatives are $\dot{x}_p = C\cosh t + D\sinh t + Ct\sinh t + Dt\cosh t$ and $\ddot{x}_p = 2C\sinh t + 2D\cosh t + Ct\cosh t + Dt\sinh t$. Substituting into the ode, we get $2C\sinh t + 2D\cosh t = \cosh t$. Therefore $C = 0$ and $D = 1/2$. The general solution is therefore

$$\begin{aligned} x(t) &= A\cosh t + B\sinh t + \frac{1}{2}t\sinh t, \\ \dot{x}(t) &= A\sinh t + B\cosh t + \frac{1}{2}\sinh t + \frac{1}{2}t\cosh t. \end{aligned}$$

With $x(0) = 0$ and $\dot{x}(0) = 0$, we obtain $A = 0$ and $B = 0$. Therefore, $x(t) = \frac{1}{2}t\sinh t$.

3. c. The fully dimensional RLC circuit equation, mass on a spring equation, and low amplitude pendulum equation are given by

$$\begin{aligned} L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q &= \mathcal{E}_0\cos\omega t, \\ m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx &= F_0\cos\omega t, \\ ml\frac{d^2\theta}{dt^2} + cl\frac{d\theta}{dt} + mg\theta &= F_0\cos\omega t. \end{aligned}$$

Comparing the three equations, it is observed that the resistance R in the RLC circuit equation plays a role analgous to the frictional coefficent c in the mass on a spring equation, and the pendulum equation.

Solutions to the Problems for Lecture 29

1. $F(s) = \int_0^\infty e^{-st}\sin bt\,dt$. Integrate by parts letting $u = \sin bt$ and $dv = e^{-st}dt$ to obtain $F(s) = \frac{b}{s}\int_0^\infty e^{-st}\cos bt\,dt$. Integrate by parts a second time letting $u = b\cos bt$ and $dv = (1/s)e^{-st}dt$ to obtain $F(s) = \frac{b}{s^2} - \frac{b^2}{s^2}F(s)$. Solving for $F(s)$, we obtain $F(s) = \frac{b}{s^2+b^2}$.

Solutions to the Problems for Lecture 30

1. The Laplace transform of the ode is given by

$$a\left(s^2X - sx_0 - u_0\right) + b\left(sX - x_0\right) + cX = G.$$

Solving for $X = X(s)$, we obtain

$$X(s) = \frac{G(s) + (as+b)x_0 + au_0}{as^2 + bs + c}.$$

Solutions to the Problems for Lecture 31

1. The Laplace transform of the ode is

$$s^2X + 5sX + 6X = \frac{1}{s+1},$$

with solution

$$X(s) = \frac{1}{(s+1)(s+2)(s+3)}.$$

A partial fraction expansion of $X = X(s)$ is written as

$$\frac{1}{(s+1)(s+2)(s+3)} = \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{s+3},$$

where the unknown coefficients can be found by using the cover-up method. Multiply both sides by $(s+1)$ and set $s = -1$ to find $a = 1/2$. Multiply both sides by $s+2$ and set $s = -2$ to find $b = -1$. Multiply both sides by $s+3$ and set $s = -3$ to find $c = 1/2$. Proceeding to take the inverse Laplace transform using the table in Appendix F, we obtain

$$x(t) = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}.$$

Solutions to the Practice quiz: The Laplace transform method

1. b. From line 9 of the Table of Laplace Transforms, we have $a = -1$ and $b = \pi$. The Laplace transform is then read off the table as $X(s) = \dfrac{s+1}{(s+1)^2 + \pi^2}$.

2. c. Use lines 16, 15, and 3 of the Table of Laplace Transforms to take the Laplace transform of the ode: $s^2X + sX - 6X = \dfrac{1}{s+1}$. Solve for $X = X(s)$ to find $X(s) = \dfrac{1}{(s+1)(s-2)(s+3)}$.

3. c. $X(s) = \dfrac{1}{(s+1)(s+2)(s+3)} = \dfrac{a}{s+1} + \dfrac{b}{s+2} + \dfrac{c}{s+3}$. Using the cover-up method, we find $a = 1/2$, $b = -1$, $c = 1/2$. Then using line 3 of the Table of Laplace Transforms, we obtain $x(t) = \dfrac{1}{2}e^{-t} - e^{-2t} + \dfrac{1}{2}e^{-3t}$.

Solutions to the Problems for Lecture 32

1. The step-down function is given by

$$1 - u_c(t) = \begin{cases} 1, & t < c; \\ 0, & t \geq c. \end{cases}$$

The step-up, step-down function, with $a < b$, is given by

$$u_a(t) - u_b(t) = \begin{cases} 0, & t < a; \\ 1, & a \leq t < b; \\ 0, & t \geq b. \end{cases}$$

2. The Laplace transform is

$$\begin{aligned} \mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt \\ &= \int_c^\infty e^{-st}f(t-c)\,dt \\ &= \int_0^\infty e^{-s(t'+c)}f(t')\,dt' \\ &= e^{-cs}\int_0^\infty e^{-st'}f(t')\,dt' \\ &= e^{-cs}F(s), \end{aligned}$$

where we have changed variables to $t' = t - c$.

3.

a)

$$\begin{aligned} f(t) &= \begin{cases} t, & \text{if } t < 1; \\ 1, & \text{if } t \geq 1 \end{cases} \\ &= t - u_1(t)\,(t-1)\,. \end{aligned}$$

b) The Laplace transform is given by $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)\,(t-1)\}$. The

Laplace transforms are found in Appendix F. We have

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \text{(using line 4 with } n = 1\text{)}$$

$$\mathcal{L}\{u_1(t)\,(t-1)\} = e^{-s}\mathcal{L}\{t\} = \frac{e^{-s}}{s^2}.$$

(using line 13 with $c = 1$ and line 4 with $n = 1$)

The result is $F(s) = \dfrac{1-e^{-s}}{s^2}$.

Solutions to the Problems for Lecture 33

1. For $a > 0$, we have

$$\int_{-\infty}^{\infty} f(x)\delta(ax)\,dx = \frac{1}{a}\int_{-\infty}^{\infty} f(x'/a)\delta(x')\,dx' = \frac{1}{a}f(0) = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\delta(x)\,dx.$$

Therefore $\delta(ax) = \dfrac{1}{a}\delta(x)$. For $a < 0$, the limits of integration reverse upon the substitution, and one finds $\delta(ax) = -\dfrac{1}{a}\delta(x)$. The general result is $\delta(ax) = \dfrac{1}{|a|}\delta(x)$.

2. The integral given by $\displaystyle\int_{-\infty}^{x} \delta(x'-c)\,dx'$ is zero if $x < c$ and one if $x > c$. This is the definition of $u_c(x)$.

3. The derivative of both sides of $u_c(x) = \displaystyle\int_{-\infty}^{x} \delta(x'-c)\,dx'$ with respect to x, using the fundamental theorem of calculus, results in $\dfrac{d}{dx}u_c(x) = \delta(x-c)$. This is only a symbolic expression because technically you can not take the derivative of a discontinuous function.

Solutions to the Problems for Lecture 34

1. We compute $x(1)$ when the Heaviside function is either zero or one. In both cases we find

$$x(1) = \frac{1}{2} - \frac{1}{e} + \frac{1}{2e^2}$$

so that $x = x(t)$ is continuous at $t = 1$.

2. The Laplace transform is

$$s^2X(s) + X(s) = \frac{1}{s}\left(1 - e^{-2\pi s}\right),$$

with solution for $X = X(s)$ given by

$$X(s) = \frac{1 - e^{-2\pi s}}{s(s^2+1)}.$$

Defining

$$F(s) = \frac{1}{s(s^2+1)},$$

and using the table in Appendix F, the inverse Laplace transform of $X(s)$ is

$$x(t) = f(t) - u_{2\pi}(t)f(t-2\pi),$$

where $f(t)$ is the inverse Laplace transform of $F(s)$. A partial fraction expansion yields

$$F(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1},$$

and the inverse Laplace transform from the table in Appendix F is $f(t) = 1 - \cos t$. We find

$$x(t) = 1 - \cos t - u_{2\pi}(t)\left(1 - \cos\left(t - 2\pi\right)\right),$$

or more clearly,

$$x(t) = \begin{cases} 1 - \cos t, & \text{if } t < 2\pi; \\ 0, & \text{if } t \geq 2\pi. \end{cases}$$

Solutions to the Problems for Lecture 35

1. Taking the Laplace transform of the ode using the table in Appendix F, we have

$$s^2X + X = 1 - e^{-2\pi s},$$

with solution for $X = X(s)$ given by

$$X(s) = \frac{1 - e^{-2\pi s}}{(s^2+1)}.$$

Defining

$$F(s) = \frac{1}{s^2+1},$$

the table in Appendix F yields $f(t) = \sin t$, and the inverse Laplace transform of $X(s)$ is

$$x(t) = f(t) - u_{2\pi}(t)f(t-2\pi) = \sin t - u_{2\pi}(t)\sin(t-2\pi),$$

or more clearly,

$$x(t) = \begin{cases} \sin t, & \text{if } t < 2\pi; \\ 0, & \text{if } t \geq 2\pi. \end{cases}.$$

Solutions to the Practice quiz: Discontinuous and impulsive inhomogeneous terms

1. a. Use the negative of a step-up, step-down function between $t = 1$ and 2, and a step-up, step-down function between $t = 3$ and 4. We have $x(t) = -(u_1(t) - u_2(t)) + (u_3(t) - u_4(t)) = -u_1(t) + u_2(t) + u_3(t) - u_4(t)$.

2. d. Laplace transform the ode to obtain $s^2X - s + X = \dfrac{1}{s} - \dfrac{e^{-2\pi s}}{s}$. Solve for $X = X(s)$ to obtain $X(s) = \dfrac{s}{s^2+1} + \dfrac{(1-e^{-2\pi s})}{s(s^2+1)}$. Write $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$, so that $X(s) = \dfrac{1}{s} - e^{-2\pi s}\left(\dfrac{1}{s} - \dfrac{s}{s^2+1}\right)$. Inverse Laplace transform to find
$$x(t) = 1 - u_{2\pi}(t)\left(1 - \cos(t - 2\pi)\right) = \begin{cases} 1, & \text{if } t < 2\pi; \\ \cos t, & \text{if } t \geq 2\pi. \end{cases}$$

3. c. Laplace transform the ode to obtain $s^2X - s + X = 1 - e^{-2\pi s}$. Solve for $X = X(s)$ to obtain $X(s) = \dfrac{s}{s^2+1} + \dfrac{1}{s^2+1} - \dfrac{e^{-2\pi s}}{s^2+1}$. Inverse Laplace transform to find
$$x(t) = \cos t + \sin t - u_{2\pi}(t)\sin(t - 2\pi) = \begin{cases} \cos t + \sin t, & \text{if } t < 2\pi; \\ \cos t, & \text{if } t \geq 2\pi. \end{cases}$$

Solutions to the Problems for Lecture 36

1. Try
$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$
and obtain
$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$
In the first sum, shift the summation index downward by two, and then combine sums to obtain
$$\sum_{n=0}^{\infty}\Big((n+2)(n+1)a_{n+2} - a_n\Big)x^n = 0.$$
We therefore obtain the recursion relation
$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \ldots.$$
Even and odd coefficients decouple, and we obtain two independent sequences starting with first term a_0 or a_1. Developing these sequences, we have for the first sequence,
$$a_0, \quad a_2 = \frac{1}{2}a_0, \quad a_4 = \frac{1}{4\cdot 3}a_2 = \frac{1}{4!}a_0,$$

and so on; and for the second sequence,

$$a_1, \quad a_3 = \frac{1}{3 \cdot 2} a_1, \quad a_5 = \frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1,$$

and so on. Using the principle of superposition, the general solution is therefore

$$\begin{aligned} y(x) &= a_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + a_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) \\ &= a_0 \cosh x + a_1 \sinh x, \end{aligned}$$

where

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Solutions to the Problems for Lecture 37

1. Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation, we have

$$\begin{aligned} y'' + xy' - y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \big((n+2)(n+1)a_{n+2} + (n-1)a_n\big)x^n = 0. \end{aligned}$$

The recursion relation is

$$a_{n+2} = -\frac{(n-1)}{(n+2)(n+1)} a_n.$$

Starting with a_0, we find $a_2 = a_0/2$, $a_4 = -a_0/24$, and $a_6 = a_0/240$. Starting with a_1, we find $a_3 = 0$, $a_5 = 0$, etc. The general solution to order x^6 is given by

$$y(x) = a_0 \left(1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{240} - \dots\right) + a_1 x.$$

Solutions to the Problems for Lecture 38

1. The general solution of the Airy's equation is given by

$$y(x) = a_0 y_0(x) + a_1 y_1(x).$$

Applying the initial conditions and the known values $y_0(0) = 1$, $y_0'(0) = 0$, $y_1(0) = 0$, and $y_1'(0) = 1$, we have

$$y(0) = 1 = a_0, \quad y'(0) = 1 = a_1.$$

Therefore, the solution is

$$y(x) = y_0(x) + y_1(x).$$

Solutions to the Practice quiz: Series solutions

1. c. Using the definitions of the hyperbolic functions, we have
$\cosh^2 t - \sinh^2 t = \dfrac{(e^x + e^{-x})^2}{4} - \dfrac{(e^x - e^{-x})^2}{4}$
$= \dfrac{1}{4}\left(e^{2x} + e^{-2x} + 2\right) - \dfrac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) = \dfrac{1}{2} + \dfrac{1}{2} = 1.$

2. d. Substituting $y = \sum\limits_{n=0}^{\infty} a_n x^n$ into the differential equation, we have
$y'' + x^2 y = \sum\limits_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum\limits_{n=0}^{\infty} a_n x^{n+2} = \sum\limits_{n=-2}^{\infty} (n+4)(n+3)a_{n+4}x^{n+2} + \sum\limits_{n=0}^{\infty} a_n x^{n+2} = 0.$
Rewrite as $2a_2 + 6a_3 x + \sum\limits_{n=0}^{\infty} \left[(n+4)(n+3)a_{n+4} + a_n\right] x^{n+2} = 0$. Setting coefficients of powers of x equal to zero results in $a_2 = a_3 = 0$, $a_{n+4} = -\dfrac{a_n}{(n+4)(n+3)}$. Starting with a_0, we find $a_4 = -a_0/12$. Starting with a_1, we find $a_5 = -a_1/20$. Since $a_2 = a_3 = a_6 = a_7 = 0$, the general solution with terms up to x^5 is $y(x) = a_0\left(1 - \dfrac{x^4}{12} + \dots\right) + a_1\left(x - \dfrac{x^5}{20} + \dots\right)$.

3. d. Substituting $y = \sum\limits_{n=0}^{\infty} a_n x^n$ into the differential equation, we have $y'' - xy' + y =$ $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\Big((n+2)(n+1)a_{n+2} + (1-n)a_n\Big)x^n$ $= 0$. The recursion relation is $a_{n+2} = \dfrac{(n-1)}{(n+2)(n+1)}a_n$. Starting with a_0, we find $a_2 = -a_0/2$, $a_4 = -a_0/24$. Starting with a_1, we find $a_3 = 0$, $a_5 = 0$, etc. The general solution up to terms proportional to x^4 is given by $y(x) = a_0\left(1 - \dfrac{x^2}{2} - \dfrac{x^4}{24} - \dots\right) + a_1 x$.

Solutions to the Problems for Lecture 39

1. The odes in matrix form are given by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and the characteristic equation of the matrix is given by

$$\lambda^2 - (a+b)\lambda + ab - c^2 = 0.$$

The roots of the characteristic equation are

$$\lambda_{\pm} = \frac{a+b \pm \sqrt{(a+b)^2 - 4ab + 4c^2}}{2} = \frac{a+b \pm \sqrt{(a-b)^2 + 4c^2}}{2}.$$

Since the discriminant is non-negative, both roots are real (and distinct provided $a \neq b$ or $c \neq 0$).

Solutions to the Problems for Lecture 40

1. With $A = \begin{pmatrix} 0 & -1 \\ -2 & -1 \end{pmatrix}$, the characteristic equation is $\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$, with roots $\lambda_1 = 1$ and $\lambda_2 = -2$. The corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The general solution is $x = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, or $x_1 = c_1 e^t + c_2 e^{-2t}$ and $x_2 = -c_1 e^t + 2c_2 e^{-2t}$.

Solutions to the Problems for Lecture 41

1. With $A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, the characteristic equation is $\lambda^2 - 2\lambda + 3 = 0$, with root $\lambda = 1 + i\sqrt{2}$ and its complex conjugate. The corresponding eigenvectors are $v = \begin{pmatrix} 2i \\ \sqrt{2} \end{pmatrix}$ and its complex conjugate. The general solution is constructed from the linearly independent solutions $X_1 = \mathrm{Re}\left\{ve^{\lambda t}\right\}$ and $X_2 = \mathrm{Im}\left\{ve^{\lambda t}\right\}$, and is

$$x = e^t \left(A \begin{pmatrix} -2\sin(\sqrt{2}\,t) \\ \sqrt{2}\cos(\sqrt{2}\,t) \end{pmatrix} + B \begin{pmatrix} 2\cos(\sqrt{2}\,t) \\ \sqrt{2}\sin(\sqrt{2}\,t) \end{pmatrix} \right),$$

or

$$x_1 = e^t \left(-2A\sin(\sqrt{2}\,t) + 2B\cos(\sqrt{2}\,t) \right), \qquad x_2 = e^t \left(\sqrt{2}A\cos(\sqrt{2}\,t) + \sqrt{2}B\sin(\sqrt{2}\,t) \right).$$

Solutions to the Practice quiz: Systems of differential equations

1. c. The system of equations in matrix form is $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. The characteristic equation of the matrix is given by $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$. The second-order ode with the same characteristic equation is given by $\ddot{x} - (a+d)\dot{x} + (ad - bc)x = 0$.

2. b. The system of equations in matrix form is $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Try $x = ve^{\lambda t}$ to obtain the eigenvalue problem $Av = \lambda v$. The characteristic equation of A from $\det(A - \lambda I) = 0$ is $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$, with roots $\lambda_1 = 3$ and $\lambda_2 = -1$. The eigenvectors are found by solving

$(A - \lambda_i I)v_i = 0$, and we find that $\lambda_1 = 3$ has eigenvector $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -1$ has eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The general solution is $x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$, and the components are $x_1 = c_1 e^{3t} + c_2 e^{-t}$ and $x_2 = c_1 e^{3t} - c_2 e^{-t}$.

3. d. The system of equations in matrix form is $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Try $\mathrm{x} = \mathrm{v}e^{\lambda t}$ to obtain the eigenvalue problem $\mathrm{Av} = \lambda\mathrm{v}$. The characteristic equation of A from $\det(\mathrm{A} - \lambda\mathrm{I}) = 0$ is $\lambda^2 + 4\lambda + 5 = 0$, with roots $\lambda = -2 + i$ and its complex conjugate. The eigenvector associated with λ is found by solving $(\mathrm{A} - \lambda\mathrm{I})\mathrm{v} = 0$, and we find that $\mathrm{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$. The general solution is $\mathrm{x} = e^{-2t}\left(A \,\mathrm{Re}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it}\right\} + B \,\mathrm{Im}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it}\right\}\right)$ $= e^{-2t}\left(A\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + B\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right)$, and the components are $x_1 = e^{-2t}(A\cos t + B\sin t)$ and $x_2 = e^{-2t}(-A\sin t + B\cos t)$.

Solutions to the Problems for Lecture 42

1. The nature of the fixed points are determined by the eigenvalues of the relevant matrices. We first write the odes as matrix equations, and then compute the eigenvalues.

a)

$$\dot{\mathrm{x}} = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\mathrm{x}.$$

The eigenvalues of the matrix are found from

$$0 = \det(\mathrm{A} - \lambda\mathrm{I}) = \lambda^2 + 5\lambda + 4 = (\lambda + 4)(\lambda + 1).$$

The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -1$. Since the eigenvalues are both negative, the fixed point is a stable node.

b)

$$\dot{\mathrm{x}} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathrm{x}.$$

The eigenvalues of the matrix are found from

$$0 = \det(\mathrm{A} - \lambda\mathrm{I}) = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3).$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. Since the eigenvalues are of opposite sign, the fixed point is a saddle point and is unstable.

c)

$$\dot{\mathrm{x}} = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}\mathrm{x}.$$

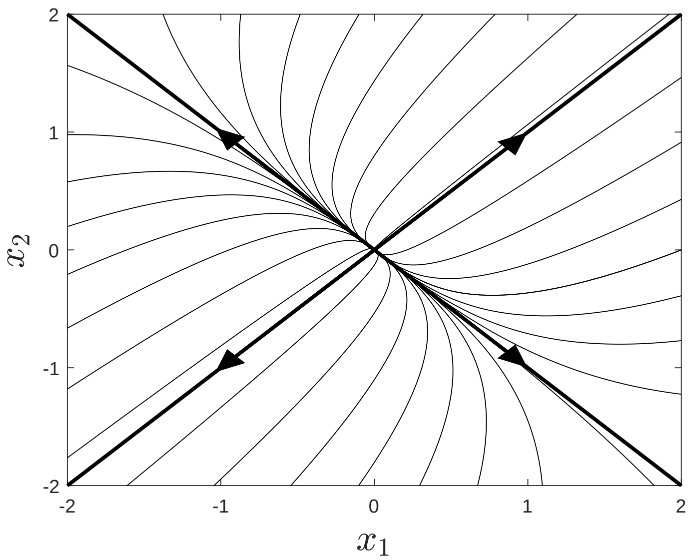
The eigenvalues of the matrix are found from

$$0 = \det(\mathrm{A} - \lambda\mathrm{I}) = \lambda^2 + \lambda + \frac{5}{4}.$$

The eigenvalues are complex and given by $\lambda = -\frac{1}{2} \pm i$. Since the real part of these complex eigenvalues is negative, the fixed point is a stable spiral point.

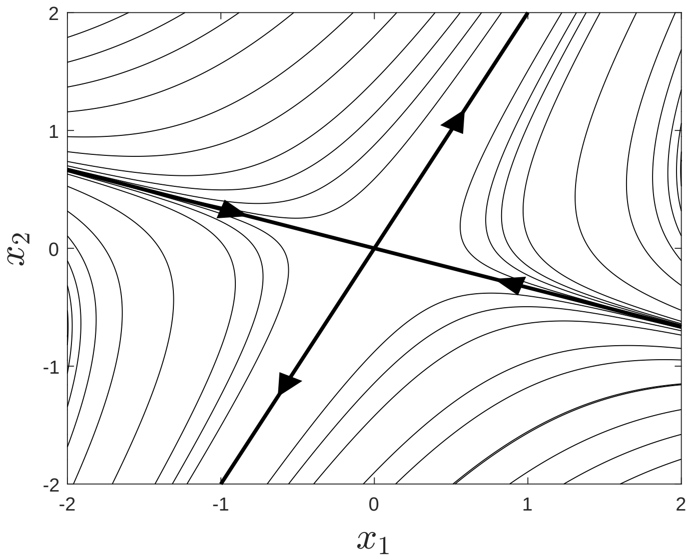
Solutions to the Problems for Lecture 43

1. With $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, the characteristic equation is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$, with roots $\lambda_1 = 3$ and $\lambda_2 = 1$. The corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The phase portrait can be sketched by drawing the lines $x_2 = x_1$ corresponding to the first eigenvector direction, and $x_2 = -x_1$ corresponding to the second eigenvector. Because both eigenvalues are positive, the motion is away from the origin and the fixed point is unstable. Also, $\lambda_1 > \lambda_2$, so the motion is faster along the first eigenvector with $x_2 = x_1$ and the curves bend in this direction. A computer-generated phase portrait is shown below.



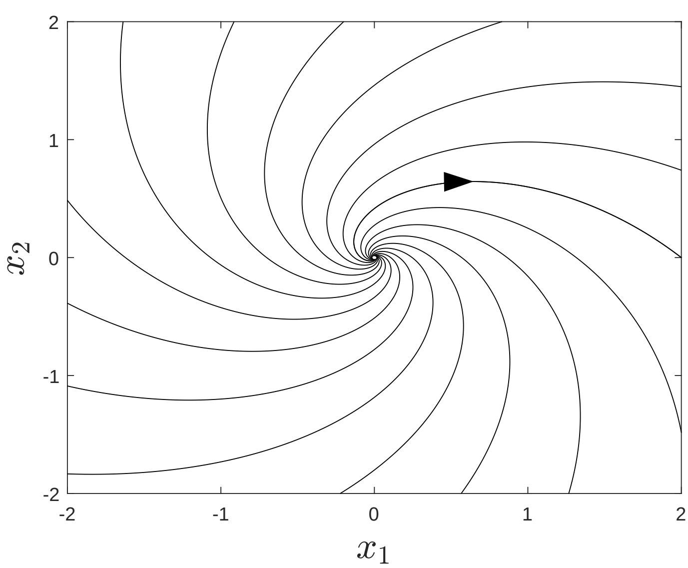
Solutions to the Problems for Lecture 44

1. With $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$, the characteristic equation is $\lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) = 0$, with roots $\lambda_1 = -2$ and $\lambda_2 = 5$. The corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -1/3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The phase portrait can be sketched by drawing the lines $x_2 = -x_1/3$ corresponding to the first eigenvector, and $x_2 = 2x_1$ corresponding to the second eigenvector. Because the eigenvalues have opposite sign, the fixed point is a saddle point. Also, $\lambda_1 < 0 < \lambda_2$ so the motion along the first eigenvector moves towards the origin, and the motion along the second eigenvector moves away from the origin. A computer-generated phase portrait is shown below.



Solutions to the Problems for Lecture 45

1. With $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, the characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$, with roots $\lambda = 1 + i$ and $\bar{\lambda}$. The corresponding eigenvectors are $\mathrm{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\bar{\mathrm{v}}$. The phase portrait is an unstable spiral point. To determine the handedness, we compute the derivatives at the point $(x_1, x_2) = (0, 1)$ and find $(\dot{x}_1, \dot{x}_2) = (1, 1)$. The stability and direction of motion indicates a clockwise spiral moving outwards. A computer-generated phase portrait is shown below.



Solutions to the Practice quiz: Phase portraits

1. a. See Problems for Lecture 43: Stable and unstable nodes.

2. b. See Problems for Lecture 44: Saddle points.

3. c. See Problems for Lecture 45: Spiral points.

Solutions to the Problems for Lecture 46

1.

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Solutions to the Problems for Lecture 48

1. The governing equation may be written as

$$\frac{m}{k}\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

With ansatz $\mathrm{x} = \mathrm{v}e^{rt}$, the matrix equation becomes $\mathrm{Av} = \lambda\mathrm{v}$, with $\lambda = mr^2/k$.The eigenvalues of A are found from $\det(\mathrm{A} - \lambda\mathrm{I}) = \lambda^2 + 3\lambda + 1 = 0$, or $\lambda_1 = -(3-\sqrt{5})/2$ and $\lambda_2 = -(3+\sqrt{5})/2$. The corresponding eigenvectors are computed to be $\mathrm{v}_1 = \begin{pmatrix} 1 & (\sqrt{5}+1)/2 \end{pmatrix}^{\mathrm{T}}$ and $\mathrm{v}_2 = \begin{pmatrix} 1 & -(\sqrt{5}-1)/2 \end{pmatrix}^{\mathrm{T}}$. The approximate values of the angular frequencies and the eigenvectors are

$$\omega_1 = 0.62\sqrt{\frac{k}{m}}, \quad \mathrm{v}_1 = \begin{pmatrix} 1 \\ 1.62 \end{pmatrix};$$

$$\omega_2 = 1.62\sqrt{\frac{k}{m}}, \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ -0.62 \end{pmatrix}.$$

Solutions to the Practice quiz: Normal modes

1. c. Newton's law and Hooke's law results in $m\ddot{x}_1 = -2kx_1 - k(x_1 - x_2)$ and $m\ddot{x}_2 = -k(x_2 - x_1)$. In matrix form, this is written as $m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3k & k \\ k & -k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

2. a. The governing equations are given by $\frac{m}{k}\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Try $\mathrm{x}(t) = \mathrm{v}e^{rt}$, and obtain $\mathrm{Av} = \lambda\mathrm{v}$, with $\lambda = \frac{mr^2}{k}$. The characteristic equation of A is

$\lambda^2 + 4\lambda + 2 = 0$, with solutions $\lambda_\pm = -2 \pm \sqrt{2}$. The angular frequencies of the normal modes are given by $\sqrt{\frac{|\lambda_\pm| k}{m}} = \begin{cases} \sqrt{\frac{(2-\sqrt{2})k}{m}} \approx 0.77\sqrt{\frac{k}{m}}, \\ \sqrt{\frac{(2+\sqrt{2})k}{m}} \approx 1.85\sqrt{\frac{k}{m}}. \end{cases}$

3. c. We find the eigenvectors of the matrix $\mathrm{A} = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$ with eigenvalues $\lambda_1 = -2 + \sqrt{2}$ and $\lambda_2 = -2 - \sqrt{2}$. We solve $(\mathrm{A} - \lambda_i \mathrm{I})\mathrm{v}_\mathrm{i} = 0$ for v_i. For λ_1, we have $\begin{pmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = 0$, or $v_{21} = (1+\sqrt{2})v_{11} \approx 2.41 v_{11}$. For λ_2, we have $\begin{pmatrix} -1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{pmatrix} \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = 0$, or $v_{22} = (1-\sqrt{2})v_{12} \approx -0.41 v_{12}$. Therefore, $\mathrm{v}_1 = \begin{pmatrix} 1 \\ 2.41 \end{pmatrix}$ and $\mathrm{v}_2 = \begin{pmatrix} 1 \\ -0.41 \end{pmatrix}$.

Solutions to the Problems for Lecture 49

1.

a) Computing $f(x+2L)$, we have

$$\begin{aligned} f(x+2L) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi(x+2L)}{L} + b_n \sin \frac{n\pi(x+2L)}{L} \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{n\pi x}{L} + 2n\pi \right) + b_n \sin \left(\frac{n\pi x}{L} + 2n\pi \right) \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \\ &= f(x). \end{aligned}$$

b) Computing the average value of $f(x)$, we have

$$\begin{aligned} \langle f(x) \rangle &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\ &= \frac{1}{2L} \int_{-L}^{L} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \left(\frac{a_n}{2L} \int_{-L}^{L} \cos \frac{n\pi x}{L} dx + \frac{b_n}{2L} \int_{-L}^{L} \sin \frac{n\pi x}{L} dx \right) \\ &= \frac{a_0}{2}. \end{aligned}$$

Therefore, $a_0 = 2 \langle f(x) \rangle$.

Solutions to the Problems for Lecture 50

1. If $f(x)$ is an odd function, then its Fourier sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

and $f(0) = \sum_{n=1}^{\infty} b_n \sin 0 = 0$. If $f(x)$ is an even function, then its Fourier cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

and

$$f'(x) = -\frac{\pi}{L} \sum_{n=1}^{\infty} n a_n \sin \frac{n\pi x}{L}.$$

Therefore, $f'(0) = -\frac{\pi}{L} \sum_{n=1}^{\infty} n a_n \sin 0 = 0$.

Solutions to the Problems for Lecture 51

1. The function $f(x)$ is odd and $L = \pi$. The coefficients of the Fourier sine series are given by

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx \\ &= -\frac{2}{n\pi} \cos nx|_0^{\pi} \\ &= \frac{2}{n\pi}(1 - \cos n\pi) \\ &= \frac{4}{n\pi} \times \begin{cases} 1, & \text{if } n \text{ odd;} \\ 0, & \text{if } n \text{ even.} \end{cases} \end{aligned}$$

The Fourier sine series is therefore

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right).$$

Solutions to the Practice quiz: Fourier series

1. b. $f(0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 0 + b_n \sin 0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$.

2. c. The square wave in the figure is an odd function of period 2π, with $f(x) = 1$ for $0 < x < \pi$. The Fourier sine series is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ with

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = -\frac{2\cos nx}{\pi n}\bigg|_0^{\pi} = \begin{cases} \frac{4}{\pi n}, & \text{n odd;} \\ 0, & \text{n even.} \end{cases}$$

Therefore, $f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$.

3. b. This is the function $f(x)$ of Question 2 where $f(x)$ is shifted to the left by $\pi/2$. If we denote this shifted function by $g(x)$, then $g(x) = f(x + \pi/2)$. We have

$g(x) = \frac{4}{\pi}\left(\sin(x+\pi/2) + \frac{\sin(3(x+\pi/2))}{3} + \frac{\sin(5(x+\pi/2))}{5} + \dots\right)$. With $\sin(x+\pi/2)$ $= \cos x$, $\sin(3(x+\pi/2)) = -\cos 3x$, etc., we have $g(x) = \frac{4}{\pi}\left(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots\right)$.

Solutions to the Problems for Lecture 52

1. We consider

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$$

and define the dimensionless time as $\tau = tD/L^2$ and the dimensionless position as $s = x/L$. Changing variables, the nondimensional diffusion equation becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial s^2},$$

which is an equation without any parameters.

Solutions to the Problems for Lecture 53

1. We have

$$u_x(0,t) = X'(0)T(t) = 0, \quad u_x(L,t) = X'(L)T(t) = 0.$$

Since these boundary conditions are valid for all t, we must have $X'(0) = X'(L) = 0$. These are called homogeneous Neumann boundary conditions.

Solutions to the Problems for Lecture 54

1. If $\lambda = 0$, the general solution of $X'' = 0$ is given by

$$X(x) = A + Bx,$$

and $X(0) = 0$ yields $A = 0$, and $X(L) = 0$ yields $B = 0$. If $\lambda < 0$, we write $\lambda = -\mu^2$, and determine the general solution of

$$X'' - \mu^2 X = 0$$

to be

$$X(x) = Ae^{\mu x} + Be^{-\mu x}.$$

The boundary condition at $x = 0$ results in $A + B = 0$, and the boundary condition at $X = L$ results in $Ae^{\mu L} + Be^{-\mu L} = 0$. Combining these two equations yields

$$A\left(e^{\mu L} - e^{-\mu L}\right) = 0.$$

The only solution for nonzero μ is given by $A = 0$, and since $B = -A$, there are no nontrivial solutions.

2. Nontrivial solutions only exist for $\lambda \geq 0$. If $\lambda = 0$, the general solution of $X'' = 0$ is given by

$$X(x) = A + Bx,$$

and the boundary conditions $X'(0) = X'(L) = 0$ require $B = 0$. We can write the eigenfunction associated with $\lambda = 0$ as $X_0(x) = 1$. If $\lambda > 0$, we let $\lambda = \mu^2$, and the general solution of the ode and its derivative is given by

$$X(x) = A\cos\mu x + B\sin\mu x, \qquad X'(x) = -\mu A\sin\mu x + \mu B\cos\mu x.$$

The boundary condition $X'(0) = 0$ requires $B = 0$ and the boundary condition $X'(L) = 0$ results in $\sin\mu L = 0$, with solutions given by $\mu_n = n\pi/L$, where n is a nonzero integer. The eigenvalues are therefore $\lambda_n = (n\pi/L)^2$ for $n = 1, 2, 3, \ldots$ with corresponding eigenfunctions $X_n(x) = \cos(n\pi x/L)$.

Solutions to the Practice quiz: Separable partial differential equations

1. d. The units on both sides of an equation must agree. We have $[u_t] = [D][u_{xx}]$, or $ut^{-1} = [D]ul^{-2}$, where u denotes concentration, t denotes time, and l denotes length. Therefore, $[D] = l^2t^{-1}$.

2. d. To solve $u_{tt} = c^2u_{xx}$, we substitute $u(x,t) = X(x)T(t)$ to obtain $XT'' = c^2X''T$. Separating, we obtain $X''/X = T''/c^2T = -\lambda$, where λ is the separation constant. The two differential equations can be written as $X'' + \lambda X = 0$ and $T'' + \lambda c^2T = 0$.

3. c. The differential equation $X'' + \lambda X = 0$ with boundary conditions $X(0) = 0$ and $X'(L) = 0$ has nontrivial solutions only when $\lambda > 0$. We let $\lambda = \mu^2 > 0$, and solve $X'' + \mu^2X = 0$ to find $X(x) = A\cos\mu x + B\sin\mu x$. Applying the boundary conditions, we have $X(0) = 0 = A$ and $X'(L) = 0 = \mu B\cos\mu L$. Since $B \neq 0$, we must have $\cos\mu L = 0$. The allowed values of μ are given by $\mu_n = \dfrac{(2n-1)\pi}{2L}$ for $n = 1, 2, 3, \ldots$. The eigenvalues and eigenfunctions are therefore given by $\lambda_n = \left(\dfrac{(2n-1)\pi}{2L}\right)^2$ and $X_n = \sin\left(\dfrac{(2n-1)\pi x}{2L}\right)$, for $n = 1, 2, 3, \ldots$.

Solutions to the Problems for Lecture 55

1. With the eigenvalues $\lambda_n = (n\pi/L)^2$, for $n = 0, 1, 2, 3, \ldots$, the differential equation for $T = T(t)$ becomes

$$T' + \left(n^2\pi^2D/L^2\right)T = 0,$$

which has solution proportional to

$$T_n = e^{-n^2\pi^2Dt/L^2}.$$

Therefore, with ansatz $u(x,t) = X(x)T(t)$ and eigenvalues λ_n, we conclude that the functions

$$u_n(x,t) = \cos(n\pi x/L)e^{-n^2\pi^2 Dt/L^2}$$

satisfy the diffusion equation and the spatial boundary conditions for every nonnegative integer n.

The principle of linear superposition for homogeneous linear differential equations then states that the general solution to the diffusion equation with the spatial boundary conditions is given by

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)e^{-n^2\pi^2 Dt/L^2},$$

where following the usual convention we have separated out the $n = 0$ solution. In the final step, we assume that $u(x,0) = f(x)$, where $f(x)$ is some specific function defined on $0 \leq x \leq L$. At $t = 0$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L).$$

We immediately recognize this equation as a Fourier cosine series for an even function $f(x)$ with period $2L$. A Fourier cosine series results because of the boundary condition $f'(0) = 0$. From our previous solution for the coefficients of a Fourier cosine series, we determine that

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

Solutions to the Problems for Lecture 56

1. We solve the diffusion equation with homogeneous Neumann boundary conditions, and model the initial concentration of the dye by a delta-function centered at $x = L/2$, that is, $u(x,0) = M_0\delta(x - L/2)$. The Fourier cosine series coefficients are therefore given by

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L M_0\delta(x - \frac{L}{2})\cos\frac{n\pi x}{L}\,dx \\
&= \frac{2M_0}{L}\cos(n\pi/2) \\
&= \begin{cases} 2M_0/L & \text{if } n = 0,4,8,\ldots; \\ -2M_0/L & \text{if } n = 2,6,10,\ldots; \\ 0 & \text{if } n = 1,3,5,\ldots. \end{cases}
\end{aligned}$$

The coefficients a_n are nonzero only for even values of n, so we can redefine n and write the solution for $u(x,t)$ as

$$u(x,t) = \frac{M_0}{L} + \frac{2M_0}{L}\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{2n\pi x}{L}\right)\exp\left(-\frac{4n^2\pi^2 Dt}{L^2}\right).$$

As one would expect for a pipe with closed ends, after a sufficiently long time, the mass of the dye becomes uniformly distributed across the length of the pipe.

Solutions to the Practice quiz: The diffusion equation

1. d. The solution of $T' + \lambda DT$ for $T = T(t)$ up to a multiplicative constant is $T = \exp(-\lambda Dt)$. Replacing λ by the eigenvalue $\lambda_n = (n\pi/L)^2$, we obtain the eigenfunction $T_n = \exp\left(-\frac{n^2\pi^2 Dt}{L^2}\right)$.

2. a. We have $u(x,t) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) \exp(-n^2\pi^2 Dt/L^2)$ and $u(x,0) = f(x)$. Therefore, $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)$. This is a Fourier cosine series for $f(x)$ and the coefficients are given by $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.

3. b. The solution for the concentration in a pipe with open ends is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 Dt}{L^2}\right).$$

With $u(x,0) = f(x)$, we have $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$. Here, $f(x) = M_0\delta\left(x - \frac{L}{4}\right)$. We have $b_n = \frac{2}{L}\int_0^L M_0\delta\left(x - \frac{L}{4}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2M_0}{L}\sin\left(\frac{n\pi}{4}\right)$. The leading-order term corresponds to $n = 1$ where we have $b_1 = \frac{2M_0}{L}\sin(\pi/4) = \frac{\sqrt{2}M_0}{L}$. Therefore $u(x,t) = \frac{\sqrt{2}M_0}{L}\sin\left(\frac{\pi x}{L}\right)\exp\left(-\frac{\pi^2 Dt}{L^2}\right)$.